

# IL NUOVO CIMENTO

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## La viscosità di volume - (II).

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**Riassunto.** — In questa nota si riportano misure del secondo coefficiente di viscosità effettuate per l'alcool amilico, il benzolo ed il tetracloruro di carbonio, con tubi di diverso diametro ed a varie frequenze. Il dispositivo sperimentale adoperato è lo stesso di quello descritto in una precedente nota. Pur non potendo dare valori assoluti del secondo coefficiente di viscosità, sembra però che si possa affermare, come già per l'acqua, l'esistenza di tale coefficiente per tutti e tre i liquidi studiati.

Continuando le ricerche sul secondo coefficiente di viscosità col metodo e col dispositivo sperimentale precedentemente descritto, è stato studiato <sup>(1)</sup> dopo quello dell'acqua, il comportamento dell'alcool amilico, per le due frequenze  $\nu = 1,8$  MHz e  $\nu = 3,0$  MHz e coi due tubi rispettivamente di raggio  $r = 3,0$  e  $r = 4,5$  cm.

Nella fig. 1 si riportano i risultati ottenuti, per questa sostanza, relativamente alle due frequenze con il tubo di raggio  $r = 3,0$  cm. Anche per l'alcool amilico si osservano, come è stato già notato per l'acqua, relativamente alla frequenza più elevata:  $\nu = 3,0$  MHz due rette con due distinte inclinazioni, ed anche per questo liquido valgono le considerazioni fatte per l'acqua. In fig. 2 sono riportate le curve per la  $\nu = 1,8$  e per i due tubi: anche in questo

<sup>(1)</sup> A. CARRELLI e F. CENNAMO: *Nuovo Cimento*, 11, 429 (1954).

caso i coefficienti angolari delle due rette per i due valori di  $r$ , e quindi di  $G$ , sono in buon accordo con i dati teorici.

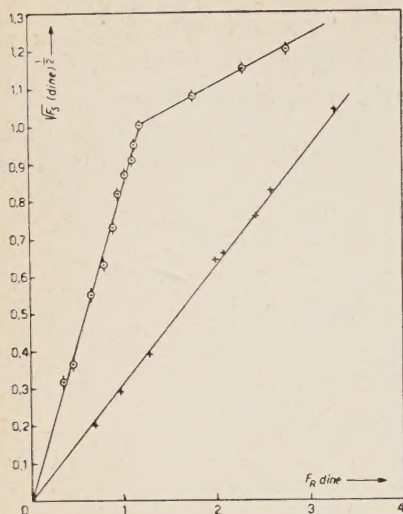


Fig. 1.

$$\text{C}_5\text{H}_{12}\text{O} \quad \left\{ \begin{array}{l} + \nu = 1,8 \text{ MHz} \\ r = 3,0 \text{ cm} \end{array} \right. \quad \left\{ \begin{array}{l} \phi \\ \phi \end{array} \right. \nu = 3,0 \text{ MHz}$$

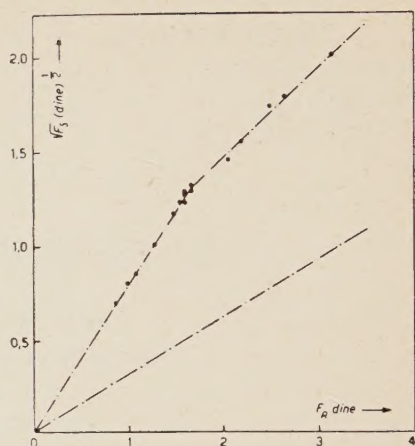


Fig. 2.

$$\text{C}_5\text{H}_{12}\text{O} \quad \left\{ \begin{array}{l} \text{---} \text{---} \text{---} r = 3,0 \text{ cm} \\ \text{---} \text{---} \text{---} \nu = 1,8 \text{ MHz} \end{array} \right. \quad \left\{ \begin{array}{l} \text{---} \text{---} \text{---} r = 4,5 \text{ cm} \\ \text{---} \text{---} \text{---} \end{array} \right.$$

Nella fig. 3 sono riportate infine le curve ottenute con il benzolo (curva 1) e il tetracloruro di carbonio, (curva 2), per la frequenza  $\nu = 1,8$  e per  $G = 0,553$ . Sullo stesso grafico sono state anche segnate le due rette relative alle stesse condizioni ( $\nu = 1,8$  MHz,

$G = 0,553$ ) già ottenute per l'acqua (curva 4) e per l'alcool amilico (curva 3).

Poichè per il gas ideale è  $\eta'/\eta = -2/3$ , l'espressione  $(2 + \eta'/\eta)$  che compare nella formula che dà la velocità del liquido <sup>(2)</sup> diventa, per tale fluido  $4/3$ ; di conseguenza in tale ipotesi per la velocità  $v$  del liquido si ottiene l'espressione:

$$(1) \quad v = \frac{\omega^2 r^2 G \cdot I}{\rho c^4} \frac{4}{3}.$$

La velocità  $v$ , quindi, se fosse lecito considerare il liquido come un gas ideale, dovrebbe, a parità di  $G$ ,  $r$ ,  $\omega$  ed  $I$  dipendere solo dai valori di  $\rho$  e  $c$ , ed essere indipendente dalla presenza o meno di un secondo coefficiente di viscosità del liquido.

<sup>(2)</sup> V. formula (2), prima nota, loc. citato. <sup>¶</sup> Nella formula (2) al secondo membro, al posto di  $\rho c$  si deve leggere:  $\rho c^4$ .

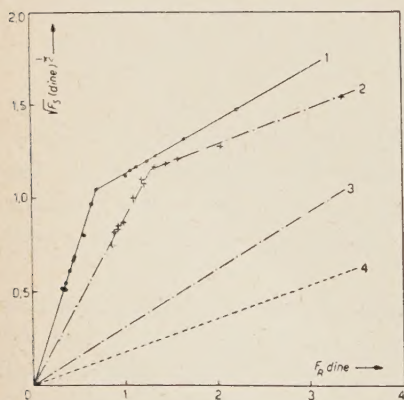


Fig. 3.



Poichè, dalle formule (3) e (4) del citato lavoro,  $v$  risulta data dalla  $v = \sqrt{F_s}/\sqrt{k}\sqrt{\varrho}$ , e l'intensità  $I$  del fascio è data dalla relazione:

$$(2) \quad I = \frac{F_R}{2\pi r^2} \frac{1}{R} \cdot c,$$

si ottiene, sostituendo tali valori nella (1):

$$(3) \quad \frac{\sqrt{F_s}}{F_R} \cdot \frac{2\pi R \sqrt{\varrho} c^3}{\omega^2 G} = \frac{4}{3} \sqrt{k}.$$

L'espressione al primo membro della (3) dovrebbe risultare cioè indipendente dal liquido adoperato, nell'eventualità che fosse lecito considerare il liquido sotto questo rispetto come un gas ideale.

Moltiplicando, cioè, i valori delle ordinate  $\sqrt{F_s}$  ottenute per ciascun liquido a parità di  $\omega^2$  e  $G$  per il corrispondente valore  $c^3\sqrt{\varrho}R$ , dipendente dal liquido e riportando i valori ottenuti in funzione delle corrispondenti  $F_R$ , dovrebbe ottenersi, qualunque sia il liquido, una retta con lo stesso coefficiente angolare.

Nella fig. 4 sono stati riportati i valori delle espressioni  $\sqrt{F_s} \cdot 2\pi R c^3 \sqrt{\varrho} / \omega^2 G$  in funzione delle  $F_R$  ottenute per il benzolo (curva 4), l'alcool amilico (curva 3), il tetracloruro di carbonio (curva 2) e l'acqua (curva 1), per la frequenza  $\nu = 1,8$  MHz e  $G = 0,553$ . Appare evidente che le rette che si ottengono presentano dei coefficienti angolari diversi, ed è da notare che la diversità riscontrata è nettamente superiore agli errori sperimentali.

Noi concludiamo questa prima serie di misure che sono evidentemente suggerite dal lavoro teorico dell'ECKART, pur riconoscendo che nelle esperienze di LIEBERMANN e nelle nostre non si è precisamente nelle condizioni volute dalla teoria, dicendo che può affermarsi che con ogni probabilità esiste nei liquidi un secondo coefficiente di viscosità.

Poichè col dispositivo da noi usato non abbiamo ancora determinato il valore del coefficiente  $k$  di proporzionalità fra la forza  $F_s$  ed il quadrato di  $v$ , possiamo dare solo i valori dei rapporti  $2 + \eta'/\eta$  relativi ai diversi liquidi che sono riportati nella tabella; da essa risulta che per il rapporto relativo ai due liquidi, benzolo e tetra-

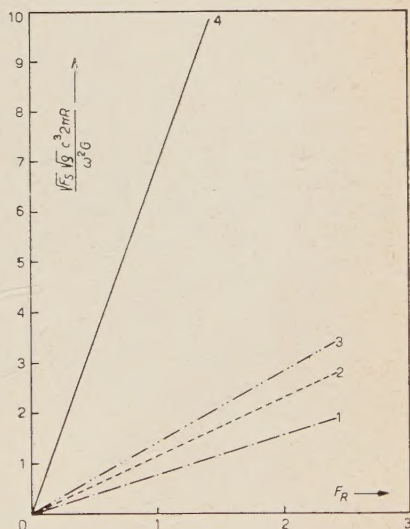


Fig. 4.

cloruro, dai nostri dati relativi alla frequenza 1,8 MHz risulta 9,0/1,4 e cioè all'incirca 6, mentre dal lavoro di LIEBERMANN lo stesso rapporto per  $\nu = 2$  MHz vale all'incirca 10 (tabella I).

TABELLA I. - Valori di  $\sqrt{k}(2 + \eta'/\eta)$ .

Liquido	$\nu=1,8$ MHz	$\nu=3,0$ MHz	$\nu=4,2$ Mz
H <sub>2</sub>	1,0	2,2	3,6
CCl <sub>4</sub>	1,4	—	—
C <sub>6</sub> H <sub>6</sub>	9,0	—	—
C <sub>5</sub> H <sub>12</sub> O	1,8	3,6	—

Si è posto:  $\sqrt{k}(2 + \eta'/\eta) = 1$  per l'acqua alla frequenza di 1,8 MHz.

Come è stato già detto per l'acqua, sembra che anche per l'alcool amilico si verifichi un aumento di  $\eta'/\eta$  all'aumentare della frequenza.

#### SUMMARY

In this note we relate on measurements of the 2<sup>nd</sup> viscosity coefficient effected for amyl alcohol, benzol and carbon tetrachloride, with tubes of different diameters and at various frequencies. We used the same experimental device described in a preceding note. Though we are not able to give absolute values of the 2<sup>nd</sup> viscosity coefficient, it could be asserted as we already did for water, that this coefficient exists for these three liquids.



## Sui contatori ad effetto Čerenkov non foccheggiati.

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**Riassunto.** — Si studiano le proprietà dei contatori ad effetto Čerenkov non foccheggiati, con particolare riguardo alle seguenti caratteristiche, in vista di eventuali applicazioni nel campo dei raggi cosmici: 1) rendimento di collezione ottica e sua uniformità; 2) tempo di risposta; 3) grande area sensibile; 4) possibilità di distinguere particelle percorrenti la stessa traiettoria in verso opposto (« proprietà antidirezionale »). Per ragionevole generalizzazione, dallo studio quantitativo di casi particolari si conclude: *a)* i contatori a pareti speculari consentono facilmente di ottenere buone « proprietà antidirezionali », ma non consentono di ottenere l'uniformità di rendimento, non appena l'area sensibile sia grande rispetto a quella del fotocatodo (§ 3); *b)* i contatori a pareti miste, opportunamente progettati (fig. 8), possono offrire buone « proprietà antidirezionali » sulla base di un rendimento basso, ma non minore di quello dei contatori speculari per quelle particelle che incidono nelle condizioni più sfavorevoli (§ 8); il cono diffondente, spesso adoperato, sembra invece ben lontano da tali risultati (§ 7); *c)* i contatori a pareti diffondenti offrono il massimo rendimento per una data area utile; l'uniformità di rendimento è anche assicurata, mentre ovviamente manca ogni « proprietà antidirezionale »; forme allungate sono sconsigliabili per tali contatori; il tempo di risposta è notevolmente maggiore per essi che per gli altri tipi, ma può ancora essere sufficientemente piccolo per molte applicazioni (v. §§ 5 e 6).

### 1. — Introduzione e posizione del problema.

Sono ormai ben note le numerosissime applicazioni dei contatori a scintillazione, che, con l'impiego di liquidi luminescenti <sup>(1)</sup>, (e, più recentemente,

(1) G. T. REYNOLDS, F. B. HARRISON e G. SALVINI: *Phys. Rev.*, **78**, 488 (1950).

anche delle plastiche luminescenti) <sup>(2)</sup> hanno ulteriormente allargato il loro, già vasto, campo di applicabilità, consentendo la costruzione di contatori molto grandi <sup>(3)</sup> <sup>(4)</sup>.

Alla tecnica relativa, sebbene giovane di pochi anni, sono già dedicati, oltre che centinaia di articoli, anche alcune ottime monografie <sup>(5)</sup>.

Ai contatori ad effetto Čerenkov <sup>(6)</sup> è stata rivolta un'attenzione molto minore; la ragione, evidentemente, è che un buon scintillatore a liquido, quando sia attraversato da una particella, sia pure al minimo di ionizzazione, produce una quantità di luce, per unità di percorso, che, come ordine di grandezza, è 100 volte maggiore di quella emessa per solo effetto Čerenkov. Quando la disposizione geometrica dell'esperienza richiede la rivelazione di particelle incidenti su un'area relativamente piccola e con una direzione ben definita (come può accadere con le macchine acceleratrici), si possono tuttavia studiare particolari dispositivi di foceggiamento che, sfruttando la caratteristica direzionalità di emissione della radiazione Čerenkov, non solo consentono di raccogliere gran parte della luce Čerenkov emessa dalle particelle che interessano, ma anche di escludere la luce emessa in altre direzioni e posizioni, con evidente ulteriore riduzione del fondo (oltre a quella risultante dall'esclusione di tutte le particelle lente e molto ionizzanti, pure peculiari ai contatori Čerenkov) <sup>(7)</sup>.

Ma anche quando non ricorrano le condizioni che consentono l'impiego di contatori «foceggiati» le caratteristiche dei contatori Čerenkov hanno trovato qualche utile applicazione, come nei problemi di albedo (v. WINCKLER <sup>(8)</sup>) e nel «heavy particle selector» di DUERDEN e HYAMS <sup>(9)</sup>, e altre riteniamo possano trovare (per esempio, nella misura della densità locale di uno sciame esteso, o nello studio dei «bursts» sotto forti spessori di Pb). Per i problemi di albedo si sfrutta la direzionalità caratteristica dell'emissione Čerenkov per distinguere due particelle relativistiche che procedano in direzioni opposte; a tale proprietà, che è esclusiva dei contatori Čerenkov, daremo il nome, in

<sup>(2)</sup> C. N. CHOU: *Phys. Rev.*, **90**, 473 (1953); ibidem, **87**, 376 e 903 (1952).

<sup>(3)</sup> J. V. JELLEY e W. J. WHITEHOUSE: *Proc. Phys. Soc.*, A **66**, 453 (1953).

<sup>(4)</sup> F. B. HARRISON: *Nucleonics*, **10**, giugno 1952, pag. 40; C. L. COWAN jr., F. REINES, F. B. HARRISON, E. C. ANDERSON e F. N. HAYES: *Phys. Rev.*, **90**, 493 (1953).

<sup>(5)</sup> S. C. CURRAN: *Luminescence and the Scintillation Counter* (London, 1953); J. B. BIRKS: *The Scintillation Counter* (London, 1953); W. H. JORDAN: in *Annual Review of Nuclear Science*, vol. I (Stanford Cal., 1952), pag. 209-266; G. A. MORTON: *The Scintillation Counter*, in *Advances in Electronics*, vol. IV, edited by L. MARTON (New York, 1952).

<sup>(6)</sup> Sulla radiazione Čerenkov e sulle sue applicazioni si veda l'eccellente rassegna di J. V. JELLEY: in *Progress of Nuclear Physics*, vol. 3 (London, 1953), pag. 84-130.

<sup>(7)</sup> Si veda, per tali tipi di contatori foceggiati, l'ottimo articolo di J. MARSHALL: *Phys. Rev.*, **86**, 685 (1952).

<sup>(8)</sup> J. R. WINCKLER: *Phys. Rev.*, **85**, 1054 (1952), cfr. anche nota <sup>(37)</sup>, cit. seconda.

<sup>(9)</sup> T. DUERDEN e B. D. HYAMS: *Phil. Mag.*, **43**, 717 (1952).



manca di un termine migliore, di *proprietà antidirezionale*, per distinguerla dalla direzionalità generica che può ottenersi in vari tipi di contatore mediante una disposizione geometrica opportuna e che in ogni caso consente di selezionare delle direzioni intorno ad una certa direzione media, ma senza possibilità di distinguere due direzioni opposte.

Nello « heavy particle selector » o per lo studio di sciame o di « bursts » sotto Pb la caratteristica saliente viene ad essere invece l'insensibilità del contatore per particelle lente, anche se molto ionizzanti.

Questa nota è dedicata allo studio sistematico dei contatori Čerenkov non focheggiati in vista di possibili applicazioni nel campo dei raggi cosmici.

\* \* \*

Precisiamo ora il nostro programma specificando le caratteristiche dei vari tipi di contatore su cui porteremo la nostra maggiore attenzione:

a) rendimento  $\eta$  di collezione ottica, cioè il rapporto tra il numero di fotoni raccolti sul fotocatodo del fotomoltiplicatore e numero di fotoni Čerenkov prodotti; è evidente, da quanto sopra si è detto, l'importanza di avere un alto valore di  $\eta$ , ma è anche evidente, per molte applicazioni, l'importanza ancora maggiore di avere un rendimento uniforme, cioè indipendente dalla traiettoria della particella (almeno entro certi limiti di « area utile » e di angolo solido);

b) tempo di risposta  $T_r$ , cioè un tempo entro il quale il contatore fornisce una buona parte dell'informazione di cui è capace; per precisare quantitativamente tale concetto occorre fare qualche convenzione; noi identificheremo  $T_r$  con il tempo di salita dell'impulso di corrente provocato dal contatore sul fotocatodo del F.M. <sup>(10)</sup>, tempo di salita definito, a sua volta, secondo la convenzione di ELMORE e SANDS <sup>(11)</sup>;

c) area utile  $A_s$  del contatore; anche questa grandezza richiederebbe qualche convenzione per una definizione precisa; riteniamo inutile farlo

<sup>(10)</sup> Si viene in tal modo a definire una caratteristica intrinseca del contatore, da cui dipende in maniera essenziale, se pur variabile da caso a caso, il potere risolutivo ottenibile dal contatore, prescindendo da eventuali ulteriori limitazioni temporali che potranno essere introdotte dai circuiti elettronici ausiliari (bisogna tener presente, tra l'altro, che il segnale prodotto da un contatore Čerenkov, essendo più piccolo, richiede una amplificazione maggiore di quello dei contatori a scintillazione) o dallo stesso fotomoltiplicatore a causa della dispersione nel tempo di transito nel F.M. stesso. Tale dispersione, per un F.M. di tipo sperimentale a 16 stadi (e struttura simile al tipo 5819), con focheggiamento è stata valutata dal MORTON (l. c. <sup>(5)</sup>) a  $\sim 4 \cdot 10^{-9}$  s e determinata sperimentalmente sullo stesso F.M., dal MORTON stesso, a  $\sim 5 \cdot 10^{-9}$  s; il BIRKS (l. c. <sup>(5)</sup>) riporta per un F.M. 1P21 (a catodo interno) il valore  $1,5 \cdot 10^{-9}$  s ottenuto da OWEN, mentre per i F.M. a persiana, tipo E.M.I. 5311 esso risulta  $\sim 3 \cdot 10^{-8}$  s (cfr. BIRKS: l. c. <sup>(5)</sup>, pag. 34, e CURRAN: l. c. <sup>(5)</sup>, pag. 66).

<sup>(11)</sup> W. ELMORE e M. SANDS: *Electronics* (New York, 1949), pag. 137.



perchè, caso per caso, sarà chiaro quale sezione del contatore potrà identificarsi con essa; inoltre non importa tanto la sua definizione precisa quanto il fatto che essa è ovviamente legata alle dimensioni geometriche del contatore in modo che, come vedremo, ogni suo aumento tende a ridurre, *caeteris paribus*, il rendimento e ad aumentare il tempo di salita.

## 2. - Schematizzazioni e notazioni base.

Poichè, come appare da quanto sopra, l'elemento fondamentale su cui verte il nostro studio, è il rendimento  $\eta$  di collezione ottica, classificheremo i contatori Čerenkov in base al modo con cui si cerca di ottenere un elevato valore di  $\eta$ , dividendoli in tre categorie, a seconda del tipo di parete usata e cioè: a) contatori a pareti speculari; b) contatori a pareti diffondenti; c) contatori a pareti miste (in parte speculari e in parte diffondenti). Per ognuna di queste tre classi considereremo prima un prototipo, per il quale svilupperemo i calcoli, nell'ambito delle schematizzazioni introdotte, in maniera il più possibile quantitativa e matematicamente rigorosa, e quindi discuteremo, talvolta in maniera necessariamente qualitativa, quali delle conclusioni ottenute per il prototipo possano considerarsi valide in generale, ed eventualmente con quali limitazioni.

Salvo esplicita avvertenza in contrario, nei calcoli successivi faremo dunque le seguenti schematizzazioni ed useremo le notazioni che seguono:

a) l'assorbimento della luce da parte del mezzo è trascurabile <sup>(12)</sup>;

b) il fotomoltiplicatore ha un catodo « esterno » circolare di area  $A_k = \pi a^2$ ; ogni fotone incidente sul fotocatodo viene da questo assorbito ed ha una probabilità  $p$  di produrre un fotoelettrone, indipendentemente dalla direzione di incidenza <sup>(14)</sup>;

<sup>(12)</sup> Tale ipotesi è praticamente soddisfatta dall'acqua finchè ci si limita alla parte visibile dello spettro e ad un cammino medio dei raggi luminosi, prima dell'incidenza nel fotomoltiplicatore, di 3 m ( $10^{-8}$  s). L'assorbimento relativo a un tale cammino, in base ai dati di ASCHINASS <sup>(13)</sup> risulta  $\sim 6\%$ . La situazione però è molto peggiore con altri liquidi o se si volesse utilizzare la parte ultravioletta dello spettro; comunque la possibilità di lunghi cammini con piccolo assorbimento può annoverarsi come uno dei vantaggi dei contatori Čerenkov sui contatori a luminescenza.

<sup>(13)</sup> E. ASCHINASS: *Ann. der Phys. und Chemie*, **65** (IX), 420 (1895).

<sup>(14)</sup> La probabilità  $p$  è in generale funzione della lunghezza d'onda, ma noi la considereremo costante e pari al valore  $p_0$  che essa ha per la lunghezza d'onda di massima sensibilità, compensando poi tale errore, nei calcoli numerici, considerando una banda di sensibilità rispetto ai numeri d'onda opportunamente ridotta. Ad esempio per un fotomoltiplicatore R.C.A. 5819 (cui ci riferiremo spesso nel seguito) il massimo di sensibilità si ha per  $\lambda_m = 4800 \text{ \AA}$  (si noti che questo punto non coincide con quello



c) la luce Čerenkov viene generata secondo le note leggi di FRANK e TAMM <sup>(15)</sup> (cfr. anche l. c. <sup>(6)</sup>) mentre la luce generata per fluorescenza è trascurabile <sup>(16)</sup>;

d) la riflessione sulle pareti speculari avviene secondo le note leggi e con *assorbimento trascurabile*; tale ipotesi non è esattamente soddisfatta, se non quando si sfrutti la riflessione totale su una superficie diottrica (il che non è sempre possibile <sup>(19)</sup>), ma, quando l'assorbimento nella riflessione sia piccolo, esso non altera sostanzialmente le conclusioni cui arriveremo, mentre non è difficile prevedere, almeno qualitativamente, le modifiche e le ulteriori limitazioni imposte dalla sua presenza;

e) la riflessione diffusa avviene secondo la nota legge del coseno; tale legge può esprimersi dicendo che la probabilità  $dP$  che un fotone, incidendo su un elemento di superficie  $d\sigma$ , venga diffuso in modo da colpire un secondo elemento  $dS$ , distante  $r$  dal  $d\sigma$ , è data da

$$(1) \quad dP/d\sigma = \cos \varepsilon \cos i / \pi r^2,$$

di massimo rendimento quantico, ma conviene riferirsi ad esso perchè è il dato più comunemente indicato nei listini); il valore corrispondente di  $p_0$  è 0,064 mentre, per integrazione sulla curva rendimento quantico-numero d'onde, si ottiene un valore medio per la banda efficace  $1,18 \times 10^4 \text{ cm}^{-1}$ . Tale semplificazione è, naturalmente, possibile, solo in quanto l'indice di rifrazione  $n$ , che determina la quantità di luce Čerenkov emessa, per unità di percorso e in un intervallo unitario di numeri d'onda, varia poco al variare di  $\lambda$ . Si noti poi che i valori da noi indicati per  $p_0$  e  $\lambda_m$  sono valori medi suscettibili di larghe fluttuazioni da un F. M. ad un altro, cfr. R. W. ENGSTROM, R. G. STODENHEIMER, and A. M. CLOVER: *Nucleonics* **10**, (Aprile 1952), pag. 58.

<sup>(15)</sup> I. FRANK e I. TAMM: *Compt. Rendus de l'Ac. Sc. U.R.S.S.*, **14**, 109 (1937).

<sup>(16)</sup> Su tale ipotesi si hanno pochi dati; tuttavia, almeno per il caso dell'acqua, i dati di BELCHER <sup>(17)</sup>, anche se ottenuti in maniera molto indiretta, sembrano offrire garanzia che la fluorescenza eventuale sia una piccola frazione della luce Čerenkov; i dati di FURST e KALLMANN <sup>(18)</sup> non possono essere usati a questo riguardo perchè, per i casi di piccola fluorescenza che a noi interessa, è difficile separare dai loro dati sperimentali, che includono anche la luce Čerenkov, la parte dovuta a tale effetto da quella dovuta a fluorescenza residua. Misure del genere sono in corso anche in questo laboratorio.

<sup>(17)</sup> E. H. BELCHER: *Proc. Roy. Soc., A* **216**, 90 (1953).

<sup>(18)</sup> H. FURST e H. KALLMANN: *Phys. Rev.*, **79**, 87 (1950).

<sup>(19)</sup> L'argomento è stato trattato, sebbene con intenti alquanto diversi, da SHURCLIFF and JONES <sup>(20)</sup>; così nel caso di un contatore parallelepipedo, un raggio di luce la cui direzione *non cada* entro tre coni di semiapertura pari all'angolo limite ed aventi le direzioni degli spigoli per assi, subisce una successione indefinita di riflessioni totali. Tale circostanza, e analoghe che si presentano per altre forme, rende ancora più difficile, a nostro parere, ottenere un contatore Čerenkov con rendimento uniforme al variare della direzione di incidenza, usando pareti riflettenti per riflessione totale quando sia grande il rapporto  $A_s : A_k$ .

<sup>(20)</sup> W. A. SHURCLIFF e R. C. JONES: *Journ. Opt. Soc. Am.*, **39**, 912 (1949).



essendo  $\varepsilon$  l'angolo tra la normale al  $d\sigma$  e la congiungente  $d\sigma$  con  $dS$  e  $i$  l'angolo relativo al  $dS$  <sup>(21)</sup>;

f) il coefficiente di riflettanza diffusa  $1 - \mu$  (o rendimento di diffusione, definito come rapporto tra flusso totale emesso e flusso totale ricevuto, o, se si vuole, come probabilità che un fotone incidente venga diffuso e non assorbito) è molto prossimo ad 1 ( $\mu \ll 1$ ) <sup>(26)</sup>;

### 3. - Cilindro a pareti speculari.

Il prototipo dei contatori a pareti speculari è costituito da un cilindro speculare, col fotocatodo al centro di una base e la base opposta nera; esso offre buone *caratteristiche antidirezionali* ed ha un tempo di risposta molto breve. La prima proprietà è diretta conseguenza della intrinseca anisotropia nell'emissione Čerenkov e per la seconda è facile vedere che il tempo di risposta risulta dell'ordine del tempo che la luce Čerenkov impiega a traversare il contatore. Basta, per fissare le idee, considerare il caso di una particella che traversi il contatore esattamente lungo l'asse; i primi fotoni che giungono sul fotocatodo sono quelli generati nelle sue immediate vicinanze e gli ultimi quelli generati presso la base opposta che, se  $h$  è l'altezza del cilindro ed  $n$  l'indice di rifrazione, impiegano un tempo  $(hn/c \cos \theta) = (hn^2\beta/c)$  (essendo  $\theta$  l'angolo del cono Čerenkov, tra traiettoria particella e raggi di luce, e  $\beta c$  la velocità della particella); siccome però tali fotoni sono stati generati prima (un tempo  $h/\beta c$ ), in definitiva il tempo di risposta risulterà circa  $(hn/c)(\beta n - 1/\beta n)$ ; ad esempio, con  $h = 30$  cm,  $n = \frac{4}{3}$ ,  $\beta \cong 1$ , si ha  $T_r \cong 0,77 \cdot 10^{-9}$  s. È chiaro che, almeno finché non si considerano particelle molto inclinate rispetto all'asse, il tempo di risposta non cambia molto al cambiare della traiettoria della particella.

<sup>(21)</sup> I piccoli scarti osservati, rispetto a tale legge, per i migliori diffusori (MgO e  $\text{MgCO}_3$ ), non sono tali da preoccupare ai nostri fini, cfr. WRIGHT <sup>(22)</sup>, HENNING e HEUSE <sup>(23)</sup>, POKROWSKI <sup>(24)</sup>, PRESTON, GORDON e SMITH <sup>(25)</sup>.

<sup>(22)</sup> H. WRIGHT: *Phil. Mag.*, **49**, 199 (1900) e *Ann. der Phys.*, **1**, 17 (1900).

<sup>(23)</sup> V. HENNING e V. HEUSE: *Zeits. f. Phys.*, **10**, 111 (1922).

<sup>(24)</sup> G. I. POKROWSKI: *Zeits. f. Phys.*, **32**, 563 (1925).

<sup>(25)</sup> J. S. PRESTON, G. W. GORDON SMITH: *Proc. Phys. Soc.*, B **65**, 76 (1952); G. W. GORDON SMITH: *Proc. Phys. Soc.*, B **65**, 275 (1952).

<sup>(26)</sup> I valori ottenuti per  $\mu$  dai vari sperimentatori variano alquanto; valori  $< 0,01$  sono stati dati da TAYLOR <sup>(27)</sup> per il  $\text{MgCO}_3$ , mentre i valori più recenti ed attendibili per il MgO, che sembra attualmente preferito, (MIDDLETON e SANDERS <sup>(28)</sup>) variano tra 0,02 e 0,03 nel visibile e restano ancora  $< 0,04$  fino a  $\lambda = 2400 \text{ \AA}$ .

<sup>(27)</sup> V. TAYLOR: *Bur. Stand. Bull.*, **16**, 421 (1920); *Journ. Opt. Soc. Am.*, **4**, 9 (1920).

<sup>(28)</sup> W. E. K. MIDDLETON e C. L. SANDERS: *Journ. Opt. Soc. Am.*, **41**, 419 (1951), dove si trova ulteriore bibliografia.



Se potessimo contentarci di un'area utile dell'ordine di quella del fotocatodo potremmo far coincidere la base con la superficie fotocatodica e si avrebbe così anche un rendimento ottico  $\eta$  uguale a 1; non appena però la superficie della base cresca e diventi grande rispetto a quella del fotocatodo dobbiamo attenderci una diminuzione di  $\eta$  e, come vedremo, una forte dipendenza di  $\eta$  dalla traiettoria della particella. Il calcolo esatto di  $\eta$  per traiettorie qualunque, è molto laborioso; noi ci limiteremo pertanto a considerare solo particelle con traiettoria parallela all'asse, considerazione che riteniamo sufficiente per trarne le seguenti conclusioni di carattere negativo:

a) non appena l'area della base sia grande rispetto a quella del fotocatodo il rendimento dipende fortemente dalla traiettoria della particella;

b) il rendimento per le particelle meno favorite è dell'ordine del (e non superiore al) rapporto *area fotocatodo:area base cilindro*.

Passiamo dunque al calcolo di  $\eta$  per una particella che attraversi il contatore parallelamente all'asse; cominciamo con l'osservare che se un raggio di luce si riflette specularmente sulla superficie cilindrica esso forma con la verticale, dopo la riflessione, lo stesso angolo di prima e che la sua proiezione sul piano di base segue pure, rispetto alla circonferenza contorno, le ordinarie leggi della riflessione.

Da questa osservazione discende intanto che se un raggio ha inizialmente una proiezione che non taglia la superficie del fotocatodo, esso non potrà mai, per successive riflessioni, incidere su questo (e ciò resta vero anche se le due basi fossero speculari e quindi il raggio di luce continuasse ad essere riflesso indefinitamente). Inoltre la proiezione di un generico raggio, nelle successive riflessioni, descrive tante corde del cerchio di base tutte uguali e, perciò, ugualmente distanti dal centro. Ne segue che, ai fini dell'incidenza del raggio sul fotocatodo, tutto succede come se la prima di queste corde, quella che passa per la proiezione  $P$  della traiettoria della particella (parallela all'asse), si riflettesse su se stessa.

Ciò premesso, distinguiamo (fig. 1) il caso che la traiettoria della particella intersechi la base in un punto  $P_1$  interno al fotocatodo da quello in cui tale intersezione ( $P_2$ ) sia esterna.

Nel primo caso il piano verticale per il raggio interseca la base e quindi il raggio incontrerà o meno il fotocatodo a seconda che, per l'altezza iniziale di emissione, vada a terminare nel tratto  $AA'$  (della corda  $BB'$ , proiezione del raggio) interno al fotocatodo oppure no.

Indicando con  $\varphi$  l'angolo tra  $P_1O$  e  $B'P_1$  (cioè tra il piano contenente la particella e l'asse e quello contenente la particella e il raggio) è chiaro che la probabilità che il fotone venga emesso in un intorno  $+d\varphi$  di tale piano è

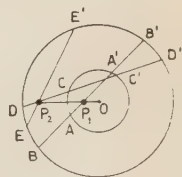


Fig. 1.

$d\varphi/2\pi$ ; a seconda dell'altezza di emissione il fotone verrà poi catturato o meno dal fotocatodo; se però il contatore è abbastanza lungo potremo ragionevolmente pensare che, *in media*, la probabilità che esso venga catturato è data da  $AA':BB'$ , cioè, indicando con  $m$  la distanza  $OP_1$ , con  $r$  il raggio del cerchio interno (che può rappresentare il fotocatodo, ma può essere anche generico) e con  $R$  il raggio di base, da  $\sqrt{r^2 - m^2 \sin^2 \varphi} : \sqrt{R^2 - m^2 \sin^2 \varphi}$ ; perciò in definitiva il rendimento  $\eta$  sarà dato da

$$(2a) \quad \eta = \frac{2}{\pi} \int_0^{\pi/2} \frac{\sqrt{r^2 - m^2 \sin^2 \varphi}}{\sqrt{R^2 - m^2 \sin^2 \varphi}} d\varphi \quad (\text{se } m \leq r).$$

Analogamente nel caso che  $P_2$  sia esterno al cerchio di raggio  $r$  si trova

$$(2b) \quad \eta = \frac{2}{\pi} \int_0^{\arcsen(r/m)} \frac{\sqrt{r^2 - m^2 \sin^2 \varphi}}{\sqrt{R^2 - m^2 \sin^2 \varphi}} d\varphi \quad (\text{con } m \geq r).$$

Il calcolo di questi integrali si riconduce ad integrali ellittici completi di 3ª specie e può esprimersi mediante funzioni tabulate, utilizzando la ben nota formula di LEGENDRE <sup>(29)</sup>, così

$$(3a) \quad \eta = \frac{2r}{\pi R} \left\{ \frac{1 - h^2}{\sin \beta} \mathbf{K}(k) + \frac{h}{\cos \beta} \cdot \frac{\pi}{2} - \frac{h}{\cos \beta} [\mathbf{K}(k)E(k', \beta) - \mathbf{K}(k)F(k', \beta) + \mathbf{E}(k)F(k', \beta)] \right\} \quad (\text{se } m \leq r),$$

dove si è posto  $h = m/r$ ,  $\cos \beta = m/R$ ,  $k^2 = m^2(R^2 - r^2)/r^2(R^2 - m^2)$ ,  $k'^2 = 1 - k^2$ ;  $F(k, \beta)$  ed  $E(k, \beta)$  sono gli integrali ellittici di 1ª e 2ª specie (di Legendre, incompleti) rispettivamente, con modulo  $k$  e argomento  $\beta$  e, infine  $\mathbf{K}(k) = F(k, \pi/2)$ ,  $\mathbf{E}(k) = E(k, \pi/2)$ .

Nel caso  $m \geq r$  la (2b) dà invece, in maniera analoga,

$$(3b) \quad \eta = 1 - \frac{2}{\pi} [\mathbf{K}(k_1)E(k'_1, \beta_1) - \mathbf{K}(k_1)F(k'_1, \beta_1) + \mathbf{E}(k_1)F(k'_1, \beta_1)], \quad (\text{se } m \geq r),$$

con  $k_1^2 = r^2(R^2 - m^2)/m^2(R^2 - r^2) = 1/k^2$ ,  $k'_1^2 = 1 - k_1^2$ ;  $\cos \beta_1 = r/R$  e gli altri simboli come sopra.

Le formule (3), considerando  $r$  fisso ed eguale al raggio del fotocatodo  $a$ ,

<sup>(29)</sup> A. M. LEGENDRE: *Fonctions Elliptiques* (Paris, 1825), pag. 138; ovvero P. F. BYRD e M. D. FRIEDMAN: *Handbuch of Elliptic Integrals for Engineers and Physicists* (Berlin, 1954), pag. 225 e segg.



mostrano che il rendimento di collezione ottica, al variare della distanza  $m$  tra la traiettoria della particella e l'asse del cilindro, va gradualmente diminuendo dal centro alla periferia, sì che le linee generali del suo comportamento si avranno anche considerando solo i tre punti caratteristici  $m=0$ ,  $m=r=a$ ,  $m=R$  per cui le formule si semplificano e danno

$$(3a_1) \quad \eta = a/R \quad (\text{per } m=0);$$

$$(3a_2) = (3b_1) \quad \eta = \frac{2}{\pi} \arcsin(a/R) \quad (\text{per } m=a);$$

$$(3b_2) \quad \eta = 1 - \sqrt{1 - \frac{a^2}{R^2}} \quad (\text{per } m=R).$$

Come si vede, quando si voglia ottenere, con un dato  $a$ , una superficie utile grande ( $R$  grande), occorre ridurre il rendimento di collezione ottica e rinunciare all'uniformità di rendimento, giacchè se  $a^2 \ll R^2$  il rendimento, che al centro è  $a/R$ , ai bordi ( $m=R$ ) diventa  $\cong a^2/2R^2$ .

Le conclusioni generali  $a)$  e  $b)$ , di cui all'inizio del paragrafo sono dunque, nel caso specifico, dimostrate.

#### 4. - Contatori a pareti speculari in generale.

È impossibile fare, di fronte alle infinite possibilità di varianti in tale tipo di contatore, considerazioni esatte e con carattere di assoluta generalità. Noi ci limiteremo a illustrare la plausibilità della generalizzazione delle conclusioni  $a)$  e  $b)$  del paragrafo precedente a contatori speculari generici, cercando di dimostrare la inutilità degli artifici che più spontaneamente si possono suggerire nel tentativo di eludere le conclusioni suddette.

Cominciamo col rilevare che l'uso di pareti tutte speculari non è giustificato; esso infatti fa perdere intanto la « caratteristica antidirezionale » e, per aumentare il rendimento, implica un numero notevole di riflessioni, con aumento del tempo di risposta; d'altra parte è chiaro, con riferimento al caso specifico di un contatore cilindrico, che, quand'anche mediante riflessioni multiple si riuscisse ad aumentare la probabilità che un dato raggio finisca per l'incidere sul fotocatodo, ciò potrà avvenire tutto al più per quei raggi la cui proiezione tagli, fin dal principio, la superficie del fotocatodo, in modo che la disuniformità di rendimento resta sempre alta. Più precisamente ricordando il ragionamento che ci ha condotti alle (2), si vede subito che noi possiamo sperare, tutto al più, mediante riflessioni ripetute, di ridurre la diminuzione di rendimento introdotta dal fattore  $\sqrt{r^2 - m^2} \sin^2 \varphi : \sqrt{R^2 - m^2} \sin^2 \varphi$ , scegliendo opportunamente l'altezza del contatore, in modo da portare suc-

cessivamente, dopo un intero percorso di andata e ritorno, sul fotocatodo, raggi che nelle precedenti incidenze sul fotocatodo erano invece rimasti esterni alla sua area; ma, anche ammesso che ciò sia possibile ripetutamente, per raggi incidenti diversamente e relativi a particelle con diverse traiettorie, non sarà tuttavia mai possibile, quando il punto  $P_2$  sia esterno (cfr. fig. 1) raccogliere raggi la cui proiezione non taglia l'area  $A_s$  del fotocatodo; ciò equivale a dire che, anche se nelle formule (2) noi sostituiamo il fattore  $\sqrt{r^2 - m^2 \sin^2 \varphi}$  con  $\sqrt{R^2 - m^2 \sin^2 \varphi}$  con l'unità, otterremo sempre una grande disomogeneità di rendimento, perchè per  $m \leq r$  si ottiene  $\eta = 1$  e per  $m \geq r$  si ottiene  $\eta = \arcsen(a/m)$  che, quando  $m$  sia grande, può risultare notevolmente piccolo.

Ad analoghe conclusioni si giunge considerando un contatore sferico speculare, in cui, ovviamente, *condizione necessaria*, se pur non sufficiente, perchè un raggio finisca con l'incidere sul fotocatodo, è che il piano contenente il raggio e passante per il centro tagli l'area del fotocatodo.

Il caso di un contatore parallelepipedo a facce speculari si presta anche ad analoghe considerazioni. Quello che tuttavia, in ogni caso, conta maggiormente è che, quando si rinunci a priori, come nei contatori a pareti tutte speculari, alla proprietà antidirezionale del contatore e si tolleri un allungamento del tempo di risposta, il contatore a pareti diffondenti (di cui ai §§ 5 e 6) offre una soluzione più semplice, col vantaggio di un rendimento uniforme, non minore sicuramente di quello corrispondente alle traiettorie più sfavorevoli in un contatore completamente speculare.

Un'ultima importante osservazione circa i vantaggi ottenibili in un contatore speculare, mediante l'uso di una guida di luce.

GARWIN<sup>(30)</sup> ha dimostrato che, perchè una guida abbia rendimento 1, è necessario che essa sia di sezione costante, in modo che essa può essere solo adoperata per *adattare* la superficie del fotocatodo a quella su cui incide la luce, quando le due abbiano *uguale area e diversa forma* <sup>(31)</sup>.

Si potrebbe ancora pensare che questa possibilità di distribuire la superficie del fotocatodo sull'area di base (riferiamoci per semplicità al caso del cilindro) potesse consentire sostanziali miglioramenti se non nel valore assoluto del rendimento, almeno nella sua uniformità; è invece facile vedere, sulla base dei calcoli del paragrafo precedente, che non è così. Ad esempio, supponendo di raccogliere la luce su una coroncina periferica (raggio  $R$ ) di area  $\pi a^2$  e quindi di spessore  $\sim a^2/2R$  (supponiamo ancora, per semplicità  $a^2 \ll R^2$ ) il rendimento, per una particella con  $m=0$ , sarebbe  $\sim a^2/2R^2$ ,

<sup>(30)</sup> R. L. GARWIN: *Rev. Scient. Instr.*, **23**, 755 (1952).

<sup>(31)</sup> Ciò non esclude naturalmente che particolari dispositivi ottici possano consentire di raccogliere su un'area più piccola (fotocatodo) la luce inizialmente incidente su una superficie maggiore, ma ciò può avvenire *solo per particolari traiettorie* delle particelle che provocano l'effetto Čerenkov, come appunto avviene nei contatori focheggiati.

mentre per  $m = R$  avremmo

$$\eta = \sqrt{1 - \left(R - \frac{a^2}{2R}\right)^2 : R^2} = \sqrt{\frac{4R^2a^2 - a^4}{4R^4}} \cong \frac{a}{R}.$$

Rispetto al caso con la superficie sensibile al centro si sono dunque semplicemente scambiate le parti tra i casi  $m = 0$  e  $m = R$ , ma non si è aumentato il rendimento, nè si è ridotta la disuniformità nel rendimento stesso.

Naturalmente la disuniformità di rendimento diminuisce se si considera una superficie sensibile anulare, con raggio esterno  $r_1$ , tale che  $a < r_1 < R$ , ma, come vedremo subito, non abbastanza; il raggio interno della corona circolare sarà dato, se vogliamo conservargli la superficie  $\pi a^2$ , evidentemente da  $r_2 = \sqrt{r_1^2 - a^2}$ ; il rendimento di collezione per un tale anello si calcola facilmente, per differenza, nei casi particolari  $m = 0$  e  $m = R$ , partendo dalle formule  $(3a_1)$  e  $(3b_2)$  e si trova:

$$(4) \quad \eta = \frac{r_1 - r_2}{R}, \quad (\text{se } m = 0),$$

$$(5) \quad \eta = \sqrt{1 - \frac{r_2^2}{R^2}} - \sqrt{1 - \frac{r_1^2}{R^2}}, \quad (\text{se } m = R),$$

che, nel caso praticamente più importante  $a^2 \ll R^2$ , si possono esprimere più comodamente in forma approssimata

$$(4') \quad \eta \cong \frac{a^2}{2Rr_1}, \quad (\text{se } m = 0),$$

$$(5') \quad \eta \cong \frac{a^2}{2R\sqrt{R^2 - r_1^2}}, \quad (\text{se } m = R).$$

Accanto a queste calcoleremo, a meno dei termini in  $(a^2/R^2)^2$ , anche la espressione di  $\eta$  per  $m = r_1$ , dato che per tale valore (per il caratteristico concentramento di luce nell'intorno del piede della traiettoria, come risulta dalle formule generali) il valore di  $\eta$  si può presumere vicino al massimo; il risultato del calcolo, utilizzando convenientemente le formule (3) ridotte alla loro espressione asintotica per  $m$  vicino ad  $r$ , è il seguente:

$$(6) \quad \eta = \frac{1}{\pi} \frac{a^2}{R^2} \left\{ \frac{R^2}{r_1 \sqrt{R^2 - r_1^2}} + \frac{1}{2} \frac{R^4}{r_1^2 (R^2 - r_1^2)} \left[ \ln \frac{16r_1^2 (R^2 - r_1^2)}{a^2 R^2} - 1 \right] \right\};$$

fissando ora ragionevolmente il valore di  $r_1$ , per esempio  $r_1 = R/\sqrt{2}$ , (per cui la (4') e la (5') danno lo stesso valore di  $\eta = a^2/R^2\sqrt{2}$ ), si ottiene

$$(6') \quad \eta = \frac{a^2}{R^2} \frac{2}{\pi} \ln \frac{4R^2}{a^2}.$$



Si vede quindi che la scelta (come superficie sensibile), di una corona anulare intermedia, riduce la disuniformità del rendimento (a misura della quale potrebbe prendersi il rapporto  $\eta_{\max}:\eta_{\min}$ , cioè approssimativamente nel caso nostro il rapporto tra il valore di  $\eta$  dato dalla (6') e quelli dati dalle (4') e (5')) rispetto al caso del fotocatodo centrale o periferico, ma tale disuniformità resta sempre grande e cresce, sia pur logaritmicamente, al crescere dell'area utile (ovviamente dell'ordine di  $\pi R^2$ ); inoltre il rendimento minimo è  $\cong a^2/2R^2$ , cioè ancora dell'ordine asserito nella conclusione *b*) del paragrafo precedente.

Le conclusioni negative dei casi precedenti non vengono meno neppure quando si immagini di realizzare, con evidenti difficoltà costruttive, una guida di luce che ripartisca la superficie fotocatodica in numerose sottili guide che si affaccino poi sulla superficie cilindrica di base in modo da venir ripartite in maniera sensibilmente uniforme; in tal caso, infatti, si ottiene bensì l'uniformità di rendimento, ma sulla base di un rendimento che sarà  $\cong a^2/R^2$ , corrispondente circa al rendimento minimo degli altri tipi e in conformità con la conclusione *b*) del paragrafo precedente. È poi da notare che un risultato analogo può anche ottenersi con contatori di tipo misto (cfr. § 8).

Osserviamo infine che la presenza di assorbimento sulle superfici speculari, riducendo in ogni caso il rendimento ed in maniera proporzionale al numero di riflessioni, rafforza (o almeno non infirma) la validità delle conclusioni negative di questo e del precedente paragrafo.

## 5. — Contatore a sfera diffondente.

Quando si voglia ottenere un rendimento di collezione uniforme, al variare della traiettoria della particella, vengono in considerazione i contatori a pareti diffondenti, con cui tale scopo può effettivamente essere conseguito, con ovvia ed implicita rinuncia alla « proprietà antidirezionale ». (Con essi, come vedremo, è anche possibile ottenere un buon rendimento con aree utili grandi, mentre il tempo di risposta risulta accresciuto, se pur non gravemente).

Come prototipo di contatore a pareti diffondenti, considereremo un contatore di forma sferica; tale forma, oltre a consentire un più facile calcolo quantitativo delle caratteristiche principali e una chiara comprensione degli elementi essenziali che giocano nel suo funzionamento, presenta anche interesse per eventuali esperienze di confronto con altre eseguite con camere di ionizzazione. Non sarà poi difficile vedere sotto quali condizioni i risultati quantitativi ottenuti per il contatore sferico possono considerarsi almeno qualitativamente validi con contatori generici.

Si tratta, in sostanza, della classica sfera integratrice di Ulbricht, per cui è facile verificare che, nell'ipotesi di una esatta validità della (1), una sola diffusione basta ad assicurare la perfetta isotropia della radiazione nell'interno

della sfera e quindi, *prescindendo da un'eventuale illuminazione diretta del fotocatodo*, l'assoluta uniformità del rendimento di collezione ottica. Risulta infatti dalla (1) che, se un elemento diffondente  $dS$  sulla sfera viene inizialmente illuminato, l'illuminazione da esso prodotta, dopo diffusione, su un altro elemento  $dS'$  della sfera è uguale per tutti i  $dS'$ .

Ne segue che, ad ogni diffusione, il fotocatodo raccoglie una frazione  $\varrho = A_k / 4\pi R^2$  della luce emessa dalle pareti,  $R$  essendo il raggio della sfera, mentre una frazione  $\mu$  viene assorbita dalle pareti. È chiaro quindi che, a lungo andare, il fotocatodo raccoglierà una frazione  $\varrho/(\mu + \varrho)$  della luce inizialmente prodotta; avremo cioè

$$(7) \quad \eta_{\infty} = \frac{\varrho}{\mu + \varrho}.$$

Possiamo anche dare una valutazione dell'andamento temporale del fenomeno di collezione, immaginando di distribuire uniformemente collezione e assorbimento, che hanno luogo nelle singole diffusioni, in un tempo  $t_m$  pari al tempo medio tra due diffusioni successive (per il caso della sfera, tenendo conto della (1), risulta

$$(8) \quad t_m = 4R/3c',$$

essendo  $c'$  la velocità della luce nel mezzo); avremo allora, per la quantità di luce  $Q$  presente ad un tempo  $t$ , l'equazione

$$(9) \quad dQ = -(\mu + \varrho)Q \, dt/t_m \quad (32),$$

da cui, posto

$$(9') \quad \tau = t_m/(\mu + \varrho),$$

si ricava

$$(10) \quad Q = Q_0 e^{-t/\tau} \quad (33),$$

e quindi un rendimento, al tempo  $t$ ,

$$(11) \quad \eta(t) = \frac{\varrho}{\mu + \varrho} (1 - e^{-t/\tau}),$$

(32) Anche se una parte della luce incidente sul fotocatodo venisse riflessa le cose non cambiano molto, almeno nel caso praticamente più importante  $\varrho \ll \mu$ .

(33) Qui si considera evidentemente la produzione dei fotoni simultanea, il che è giustificato dai tempi di risposta relativamente lunghi che poi si ottengono.



che, per  $t \gg \tau$  dà ovviamente la (7) e per  $t \ll \tau$  dà invece

$$(12) \quad \eta(t) = \frac{\varrho}{\mu + \varrho} \frac{t}{\tau} \left( 1 - \frac{1}{2} \frac{t}{\tau} + \frac{1}{6} \frac{t^2}{\tau^2} - \dots \right) \cong \varrho t / t_m.$$

In tale caso limite (come è ovvio, dato che per  $t \ll \tau$  si utilizza un piccolo numero di diffusioni) si ha un rendimento indipendente da  $\mu$ .

In generale però la piccola ampiezza del segnale richiederà l'utilizzazione di buona parte di esso ed allora dovremo contare su un tempo di salita dell'ordine di  $\tau$ . Qualche numero gioverà a fissare le idee sulle possibilità offerte da tali tipi di contatori; per un contatore di 30 cm di diametro (come le camere di ionizzazione usate nelle classiche esperienze di LAPP <sup>(34)</sup>, FAHY <sup>(35)</sup> e STINCHCOMB <sup>(36)</sup> sui bursts) con un solo fotomoltiplicatore 5819 (di area fotocattodica  $A_k \cong 11 \text{ cm}^2$ ) avremo  $\varrho = 3,9 \cdot 10^{-3}$ ; supponendo di poter contare su un valore di  $\mu = 3 \cdot 10^{-2}$  avremo un rendimento  $\eta = 11,5\%$  e un tempo di risposta  $T_r = \sqrt{2\pi} \tau = \sqrt{2\pi} \cdot (4/3) R : [(c/n)(\mu + \varrho)] = 6,6 \cdot 10^{-8} \text{ s}$  (con  $n = 4/3$  corrispondente all'acqua). Il segnale prodotto da una particella relativistica ( $1 - \beta \ll 1$ ) che percorresse in detto contatore 20 cm, risulterebbe di circa 34 elettroni liberati dal fotocatodo, ma anche tagliando il segnale a  $t = 10^{-8} \text{ s}$  si ottiene ancora un segnale di 11 elettroni.

È da notare che un risultato analogo al taglio dell'impulso ad un tempo  $t$  può ottenersi, oltre che con i soliti circuiti  $RC$ , anche aumentando artificialmente  $\mu$ ; in tal modo infatti si riduce il rendimento, ma anche  $\tau$ ; così nell'esempio precedente, portando  $\mu$  al valore  $9 \cdot 10^{-3}$  si ottiene ancora un segnale di 12 elettroni con un  $T_r \cong 2,4 \cdot 10^{-8} \text{ s}$ . Questa possibilità di ridurre il tempo di risposta direttamente agendo sul contatore, sia pure a scapito del rendimento, è una caratteristica interessante dei contatori Čerenkov a pareti diffondenti.

In conclusione possiamo dire che l'uso di pareti diffondenti consente di usare aree utili notevoli, con rendimento uniforme e ancora abbastanza grande (sempre alquanto maggiore, ad ogni modo, a quello ottenibile con pareti speculari, almeno per le particelle che in questi danno il rendimento minore) e con tempi di risposta molto maggiori di quelli ottenibili potenzialmente da un contatore Čerenkov, ma ancora abbastanza brevi per molti usi; inoltre, per una data superficie utile, i due requisiti di un alto rendimento e di un breve tempo di risposta, contrastanti fra loro, possono tuttavia esser facilmente variati uno a spese dell'altro, in modo da ottenere la soluzione più adatta al problema specifico.

<sup>(34)</sup> R. E. LAPP: *Phys. Rev.*, **69**, 321 (1946).

<sup>(35)</sup> E. F. FAHY: *Phys. Rev.*, **83**, 413 (1951).

<sup>(36)</sup> T. G. STINCHCOMB: *Phys. Rev.*, **83**, 422 (1951).

In ogni caso appare evidente, anche per tali contatori, l'utilità di disporre di F.M. con fotocatodo di grande superficie, di cui alcuni tipi allo stadio sperimentale sono già preannunziati <sup>(37)</sup>.

## 6. - Contatori a pareti diffondenti di forma qualunque.

Uno studio quantitativo delle caratteristiche di contatori a pareti diffondenti e di forma diversa da quella sferica va incontro a notevoli complicazioni di calcolo; d'altra parte una forma non sferica può essere suggerita od imposta da varie ragioni.

Ad esempio, quando si abbia a che fare con particelle, incidenti su una superficie abbastanza grande, ma relativamente collimate intorno ad una certa direzione, può essere opportuno che il contatore termini con due facce piane, parallele fra loro e normali a detta direzione; in tal caso si può pensare ad un contatore cilindrico. Analogamente un'esperienza in coincidenza con vassoi di contatori di Geiger, suggerisce una forma parallelepipedica per il contatore Čerenkov, come, ad esempio, quello realizzato da DUERDEN e HYAMS (l. c. <sup>(9)</sup>) nel loro « heavy particle selector », che è cubico.

Noi non entreremo nella discussione particolare di nessuna di queste forme, dato anche che le soluzioni da adottare sono in stretta relazione col resto dell'esperienza; vogliamo però osservare che *condizione essenziale perchè i vantaggi del contatore a diffusione non vadano perduti* è che esso *abbia una forma sufficientemente regolare* e che *le tre dimensioni massime del contatore non siano molto diverse*; soddisfatte tali condizioni *i risultati ottenuti per il caso della sfera* possono ritenersi *approssimativamente validi* e le corrispondenti conclusioni qualitative possono estendersi dalla sfera alle altre forme.

La cosa è abbastanza intuitiva e può essere ulteriormente precisata osservando che le proprietà fondamentali e lo stesso calcolo quantitativo del rendimento riposano essenzialmente sull'immediata isotropizzazione della radiazione luminosa che si ha in una sfera diffondente; pertanto i contatori di forma non sferica tanto più e tanto meglio si avvicineranno al comportamento già studiato del contatore sferico, quanto più sarà rapido (come tempo e come numero di diffusioni) il processo di isotropizzazione. Una volta raggiunta una distribuzione pressochè isotropa, il comportamento del contatore può ancora essere ragionevolmente descritto, anche quantitativamente, allo stesso modo che per un contatore sferico, introducendo un opportuno tempo medio  $t_m$  tra due diffusioni successive e il rapporto  $\varrho$  tra l'area del fotocatodo e l'area totale (diffondente) del contatore.

<sup>(37)</sup> B. R. LINDEN: *Nucleonics*, **11**, n. 9 (Settembre 1953), pag. 30; cfr. anche J. R. WINCKLER and K. ANDERSON: *Phys. Rev.*, **93**, 596 (1954).



Sebbene sia evidentemente difficile precisare in un dato quantitativo, valevole per ogni forma, la rapidità di isotropizzazione (sì che il problema dovrà essere studiato caso per caso), è chiaro tuttavia che le condizioni qualitative enunciate inizialmente corrispondono proprio a tale esigenza di rapida isotropizzazione. Insistiamo nel rilevare che una isotropizzazione lenta annulla i vantaggi del contatore diffondente, in quanto non consente di assicurare l'uniformità di rendimento, riduce il rendimento per il maggior numero di diffusioni necessarie, ed aumenta, per lo stesso motivo, il tempo di risposta.

La trattazione di un caso semplice varrà ad illustrare quanto seriamente una forma allungata ritardi l'isotropizzazione. Consideriamo un cilindro a pareti diffondenti, di raggio  $R$  e altezza infinita nei due sensi, in cui l'illuminazione sia inizialmente concentrata su un cilindro di altezza infinitesima; posto l'asse  $x$  coincidente con l'asse del cilindro e l'origine nel piano corrispon-

dente all'illuminazione iniziale, l'illuminazione di un elemento qualunque di superficie, dopo un numero  $n$  qualunque di diffusioni risulterà funzione solo di  $x$ , per ragioni di simmetria. Dopo la prima diffusione, ad esempio, si dimostra facilmente che la quantità di luce  $dQ$  incidente sul cilindro tra  $x$  e  $x + dx$ , normalizzata ad una quantità totale uguale a 1 è

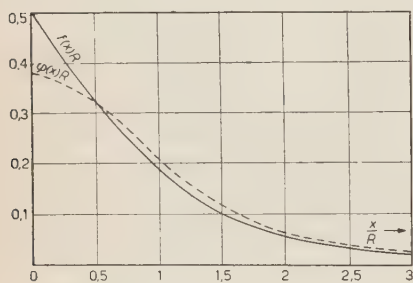


Fig. 2. - Distribuzione luminosa su un cilindro indefinito dopo una sola diffusione: — formula esatta (13); - - - - - formula approssimata (14).

$$(13) \quad dQ = f(x) dx =$$

$$= \frac{1}{2R} \left\{ 1 - \frac{|x|(6R^2 + x^2)}{(4R^2 + x^2)^{3/2}} \right\} dx,$$

che è rappresentata a tratto pieno in fig. 2.

Lo scarto quadratico medio di tale distribuzione risulta  $\bar{x} = R\sqrt{8/3}$ ; la teoria delle funzioni caratteristiche e il teorema sulla convoluzione di due distribuzioni, ci assicurano che dopo  $n$  diffusioni lo scarto quadratico risulta  $\sqrt{n} \bar{x}$ , il che dimostra già la lentezza del processo di isotropizzazione (che, naturalmente, deriva dal fatto che, ad ogni diffusione, un fotone ha uguale probabilità di esser diffuso da una parte o dall'altra).

Per il calcolo della funzione di distribuzione dopo  $n$  diffusioni, la espressione esatta (13) della distribuzione dopo una diffusione su un cilindretto infinitesimo è troppo complicata; essa tuttavia può essere sostituita, con buona approssimazione, da un'espressione del tipo

$$(14) \quad dQ = \varphi(x) dx = \frac{2a^3}{\pi} \frac{1}{(a^2 + x^2)^2} dx,$$

che meglio si presta per i calcoli ed è rappresentata a tratteggio in fig. 2, nor-

malizzata ad una quantità totale uguale a 1, come la (13). La scelta di tale forma particolare approssimata, tra le varie che potevano prestarsi al calcolo, è stata fatta per consentire una buona rappresentazione asintotica della (13) per  $x \gg R$ , il che può ottenersi esattamente dalla (14) scegliendo

$$(15) \quad a = R\sqrt[3]{3\pi/2}.$$

Una buona rappresentazione nei punti lontani ci è parsa essenziale (e ci ha fatto escludere, ad esempio, la semplice approssimazione con una gaussiana); inoltre la (14) tende a favorire, rispetto alla (13), le diffusioni lontane, e la sua varianza  $\bar{x}^2 = a^2$  è pure un po' maggiore di quella della (13); ciò nonostante, come ora vedremo, il calcolo quantitativo conferma la lentezza nell'isotropizzazione.

Infatti la funzione caratteristica relativa alla distribuzione (14) risulta

$$g(y) = \int_{-\infty}^{+\infty} \exp[-ixy] \varphi(x) dx = \exp[-a|y|](1 + a|y|).$$

Pertanto la funzione caratteristica della distribuzione  $\Phi_n(x)dx$  dopo  $n$  diffusioni sarà

$$G_n(y) = [g(y)]^n = \exp[-an|y|](1 + a|y|)^n.$$

Il calcolo di  $\Phi_n(x)$  a partire da  $G_n(y)$  è facile e dà

$$(16) \quad \begin{aligned} \Phi_n(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp[-ixy] G_n(y) dy = \\ &= \frac{\cos \theta}{\pi an} \left\{ \cos \theta + \cos 2\theta \cos \theta + \left(1 - \frac{1}{n}\right) \cos 3\theta \cos^2 \theta + \right. \\ &\quad + \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cos 4\theta \cos^3 \theta + \dots + \\ &\quad + \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{n-2}{n}\right) \cos n\theta \cos^{n-1} \theta + \\ &\quad \left. + \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{n-2}{n}\right) \frac{1}{n} \cos (n+1)\theta \cos^n \theta \right\}, \end{aligned}$$

dove si è posto

$$(16') \quad \theta = \arctg(|x|/an).$$

La fig. 3 riporta i risultati del calcolo numerico per  $n = 10$  e mostra chia-



ramente come buona parte della luce sia rimasta ancora confinata entro una distanza  $\simeq 3a$  dall'origine; per confronto abbiamo riportato anche la distribuzione gaussiana di uguale varianza; la distribuzione effettiva si può presu-

mere sia compresa fra le due. L'inopportunità di usare contatori a pareti diffondenti di forma allungata riteniamo sia così sufficientemente illustrata.

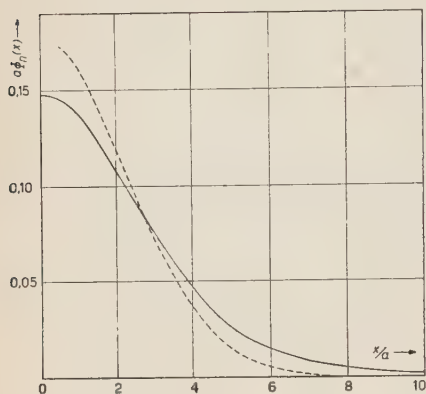


Fig. 3. - Distribuzione luminosa su un cilindro indefinito dopo 10 diffusioni (linea a tratto pieno); la linea tratteggiata rappresenta la distribuzione gaussiana di uguale varianza.

l'aggiunta di qualche parete diffondente. A tale categoria di contatori misti (pareti in parte speculari e in parte diffondenti) appartiene anche il primo contatore Čerenkov con cui JELLEY <sup>(38)</sup> ha rivelato particelle della radiazione cosmica; esso è costituito essenzialmente da un cilindro speculare, di cui una base è raccordata alla superficie fotocatodica (di area alquanto minore) mediante un cono diffondente. Un contatore del genere può ancora godere, in larga misura, della « proprietà antidi-rezionale », come i contatori speculari; può quindi essere ancora utilizzato per problemi di albedo: nel contatore di Jelley tale proprietà fu sfruttata per dimostrare la genuinità dell'effetto. Cominciamo col calcolarci la probabilità che un fotone, incidendo su un elemento  $dS$  di una superficie conica diffondente, venga diffuso dal  $dS$  su un cerchio, di raggio  $r$ , la cui circonferenza appartiene al cono (fig. 4).

Detta  $z$  la distanza del  $dS$  dal piano del cerchio, partendo dalla (1) si trova

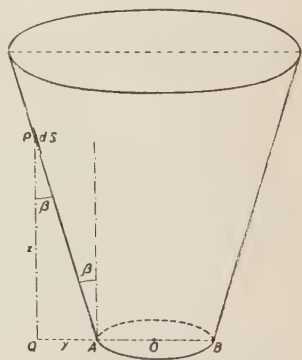


Fig. 4.

<sup>(38)</sup> J. V. JELLEY: *Proc. Phys. Soc.*, A **64**, 82 (1951); ovvero cit. <sup>(6)</sup> pag. 106 segg.

che tale probabilità è data da:

$$(17) \quad p(z) = \frac{1}{r + z \operatorname{tg} \beta} \left\{ \frac{2r^2 + 3rz \operatorname{tg} \beta + z^2(1 + \operatorname{tg}^2 \beta)}{\sqrt{4r^2 + 4rz \operatorname{tg} \beta + z^2(1 + \operatorname{tg}^2 \beta)}} - \left( r \sin \beta + \frac{z}{\cos \beta} \right) \right\}.$$

La fig. 5 dà l'andamento, per diversi valori di  $\beta$ , della  $p$ , espressa però in funzione di  $y = z \operatorname{tg} \beta$ .

Se noi identifichiamo ora il cerchio colla superficie del fotocatodo si vede subito che l'aggiunta del cono diffondente ad un contatore cilindrico speculare migliora il rendimento, ma tale miglioramento è cospicuo solo nelle vicinanze del fotocatodo, sì che non si può sperare da esso uniformità di rendimento.

Se noi infatti pensiamo di utilizzare solo la prima diffusione ciò è evidente dai grafici di fig. 5; d'altra parte, utilizzando sempre la formula (17) si può vedere facilmente che un fotone incidente sul cono diffondente ha una probabilità notevole di essere diffuso all'indietro sulla

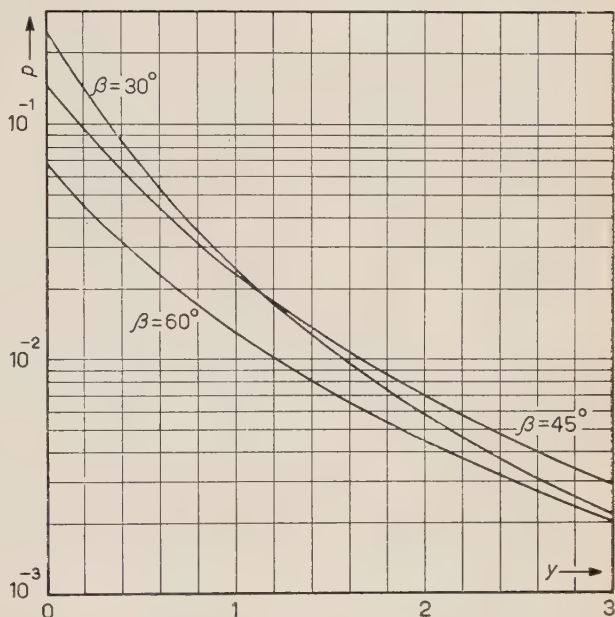
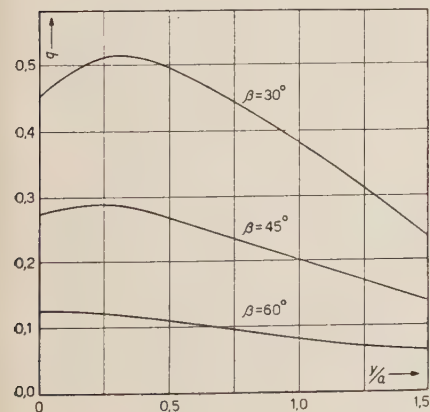


Fig. 5. — Probabilità che un fotone venga diffuso da un dato elemento di tronco di cono diffondente, direttamente sulla base minore.



base del contatore e quindi la probabilità che esso subisca numerose diffusi sul cono, prima di essere catturato dal fotocatodo è molto piccola, specie per gli elementi più lontani dal fotocatodo. Ad esempio, la fig. 6 dà la probabilità  $q(y/a)$  che un fotone inci-

Fig. — 6. Probabilità che un fotone incidente su un elemento di tronco di cono diffondente venga diffuso su altro elemento del tronco di cono stesso ( $R = 2,5a$ ).



dente su un punto del cono diffondente venga diffuso su un altro elemento del cono stesso, per un contatore avente i seguenti dati: raggio del fotocatodo  $a$ , raggio della parte cilindrica  $R = 2,5 a$ .

Quando l'uniformità di rendimento non sia un requisito essenziale e, soprattutto, quando i due raggi  $R$  del contatore ed  $a$  del fotocatodo non siano molto diversi, il contatore a cono diffondente potrà ancora essere usato e rappresenta una soluzione particolarmente semplice;

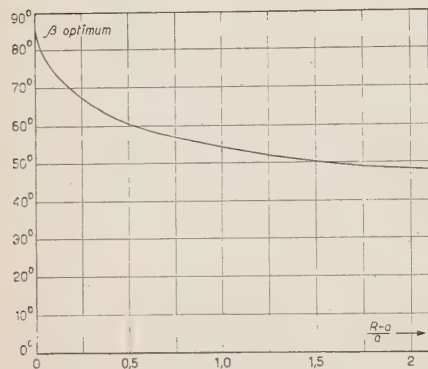


Fig. 7. - Semiapertura optimum di un tronco di cono diffondente, secondo il criterio del testo.

in tal caso ci si può domandare qual'è il valore di  $\beta$  più opportuno; l'esame dei diagrammi di fig. 5 può servire come utile guida. Tuttavia si può anche cercare qualche criterio di scelta quantitativo e, poichè  $p(z)$ , per un dato  $\beta$ , è funzione decrescente di  $z$ , come si verifica facilmente, un buon criterio può essere quello di imporre che, fermi restando  $a$  e  $R$ , sia massimo, rispetto a  $\beta$ , il valore che  $p$  assume nel punto più lontano dal fotocatodo. Ciò porta per  $\beta$  alla seguente equazione, la cui soluzione, in funzione di  $(R-a)/a = \gamma - 1$  è riportata graficamente in fig. 7:

$$(18) \quad 3 \operatorname{cosec}^6 \beta + (3 + 18\gamma - 5\gamma^2) \operatorname{cosec}^4 \beta - 4\gamma(\gamma - 1)(5\gamma + 3) \operatorname{cosec}^2 \beta + 16\gamma^2(\gamma - 1)^2 = 0.$$

## 8. - Altri tipi di contatore misto.

L'enorme varietà di tipi di contatore misto che si possono immaginare rende praticamente impossibile una discussione con carattere di assoluta generalità; se però si tengono presenti quali caratteristiche possano farlo preferire al contatore speculare o al contatore diffondente si vede che il contatore misto dovrà avere, come quello speculare, possibilità di contatori allungati e con « proprietà antidirezionali », e, in più, qualche caratteristica del contatore diffondente, cioè o un elevato rendimento o, almeno, un rendimento uniforme. Abbiamo visto che un contatore a cono diffondente non dà rendimento uniforme e nemmeno un rendimento molto più alto di quello speculare: riteniamo anche che non sia possibile ottenere un rendimento minimo elevato, comunque si vari la struttura del contatore misto. Possiamo invece puntare sull'uniformità di rendimento (naturalmente con un rendimento piccolo); tale requisito

può essere ottenuto con un contatore misto, del tipo rappresentato in fig. 8 e che si sta sperimentando in questo Laboratorio. Si tratta, in sostanza, del solito cilindro a pareti speculari, con la base superiore nera e terminante, nella parte inferiore, in una sfera diffondente alla quale si affaccia il fotomoltiplicatore K (o i fotomoltiplicatori). Eventualmente la sfera diffondente può essere separata dal cilindro speculare da un vetro smerigliato, di cui vedremo fra poco la funzione, ma da cui possiamo per ora prescindere.

Il risultato che si ottiene da un tale contatore può essere facilmente compreso, trascurando in un primo momento, l'assorbimento delle pareti diffondenti; in tal caso, infatti, la luce Čerenkov convogliata dal cilindro speculare, investe la superficie diffondente della sfera, che immediatamente la isotropizza conseguendo così una completa uniformità di rendimento (a prescindere da un'eventuale illuminazione diretta del fotocatodo, che può essere evitata); il rendimento viene però alquanto ridotto e precisamente esso vale  $\eta_{\infty} = A_k / (A_k + A_c)$  se con  $A_c = 2\pi R \sqrt{R^2 - r^2}$

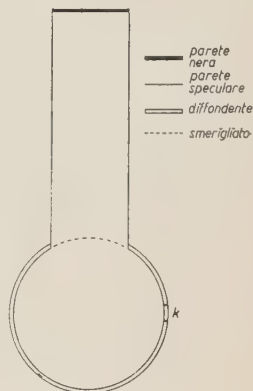


Fig. 8.

si indica la superficie sferica di raggio  $R$  (sfera diffondente) sottesa dal cerchio di base  $\pi r^2$  (corrispondente al contatore cilindrico). Tale rendimento ottico è dunque piccolo, ma dello stesso ordine di quello che si ottiene da un contatore speculare normale (v. § 3) con cui si voglia esser sicuri di non perdere particelle (regolandoci quindi sul minimo rendimento ottico), con in più il vantaggio però dell'uniformità.

Si vede poi che  $\eta_{\infty}$  può essere variato di poco (variando  $R$  e quindi  $A_c$  che praticamente oscillerà tra  $2\pi r^2$  per  $R = r$  e  $\cong \pi r^2$  per  $R \gg r$ ) e comunque non vale la pena di aumentare molto  $R$  perchè il vantaggio nel rendimento diventa piccolo e si aumenta perciò inutilmente il tempo di risposta.

Una trattazione più esatta, tenendo conto del valore finito di  $\mu$ , può farsi semplicemente utilizzando le formule (7) e (11) in cui in luogo di  $\mu$  si ponga  $\mu + (A_c/4\pi R^2)$ .

Si trova così

$$(19) \quad \eta_{\infty} = \frac{A_k}{\mu \cdot 4\pi R^2 + A_c + A_k},$$

$$(20) \quad \eta(t) = \eta_{\infty} (1 - e^{-t/\tau}),$$

$$(20') \quad \tau = t_m \cdot \frac{4\pi R^2}{\mu \cdot 4\pi R^2 + A_k + A_c}.$$

Se si vuole esaltare la « *caratteristica antidirezionale* » del contatore, non sarà opportuno che la sfera diffondente sia riempita di liquido; in tal caso la superficie di separazione tra cilindro e sfera dovrà essere smerigliata per assicurare il passaggio di una notevole quantità di luce; ciò assicura anche una prima diffusione e porta all'esclusione di eventuale luce diretta sui fotocatodi; la quantità di luce raccolta dalla sfera è ora un po' minore, ma ciò è in gran parte compensato dal fatto che nelle successive diffusioni se ne perde meno (l'area  $A_c$  non rappresenta più una parete nera, ma solo una parete parzialmente assorbente e parzialmente diffondente). Possono quindi ritenersi validi i risultati precedenti con una più netta definizione delle caratteristiche specifiche come contatore Čerenkov.

Un esempio numerico potrà illustrare meglio le possibilità di tale tipo di contatore; con riferimento alla fig. 8 sia  $h = 30$  cm l'altezza della parte cilindrica speculare, piena d'acqua,  $r = 6$  cm il raggio di tale parte,  $R = 10$  cm il raggio della sfera diffondente, separata dal cilindro da un vetro smerigliato (per cui ammetteremo una diffusione isotropa nell'angolo solido  $2\pi$ ). Si ottiene un contatore con un'area sensibile  $A_s = \pi r^2 = 113$  cm<sup>2</sup> ( $A_c = 126$  cm<sup>2</sup>) e quindi con  $\mu = 0,03$  e  $A_k = 11$  cm<sup>2</sup> (F.M. 5819 R.C.A.), un rendimento  $\eta_\infty \simeq 0,06$  secondo la (19); la presenza del vetro smerigliato riduce, come si è detto, il rendimento di poco; approssimativamente si potrà contare su un  $\eta_\infty \simeq 0,05$ ; ciò corrisponde, per una particella percorrente 30 cm in acqua, a un segnale di 22 elettroni, mentre il tempo di risposta risulta dalla (20')  $T_r = \sqrt{2\pi} \tau \simeq 0,8 \cdot 10^{-8}$  s (che sale a  $1,1 \cdot 10^{-8}$  s tenendo conto della presenza del vetro smerigliato).

Si ottiene così un contatore con rendimento sufficiente ed uniforme, con tempo di risposta abbastanza piccolo, con netta « *caratteristica antidirezionale* » e quindi di facile taratura e controllo nei riguardi della genuinità dell'effetto Čerenkov.

## SUMMARY

The properties of « non focussed » Čerenkov effect counters are investigated, especially in view of their possible applications in cosmic-ray work. Attention is mainly concentrated on optical collection efficiency and its uniformity, time of response (defined as the rise time of the electrical pulse produced on the photocathode), large sensitive area and ability to distinguish between particles travelling in exactly opposite directions (which we call « *antidirectional property* » and is peculiar to Čerenkov detectors, owing to the intrinsic anisotropy in the emission of the Čerenkov light). Čerenkov detectors are classified into three classes according to the optical properties of the walls, namely specular (§§ 3, 4), diffusing (§§ 5, 6) and mixed (§§ 7, 8). For



each class a typical geometry is discussed in detail (§§ 3, 5, 7), while reasonable generalization of results is discussed in separate paragraphs (§§ 6, 4, 8). The main conclusions are: *a*) properly designed specular detectors, while easily exhibiting good « antidirectional properties » and occasionally high collection efficiency, can hardly ensure uniformity of response with a sensitive area which be large compared to that of the photocathode; *b*) good « antidirectional properties » and uniformity of response can be best achieved with mixed counters (fig. 8), the optical efficiency being then uniformly low (not lower, however, than the efficiency of specular counters for particles impinging under the less favourable incidence); the often employed diffusing cone, however, does by no means appear to be the best solution; *c*) diffuse counters allow the largest area for a given optical efficiency and vice-versa; they ensure absolute uniformity of efficiency, but obviously do not show any « antidirectional property »; they also show considerably longer time of response, with respect to other types, but this time may be of the order of some  $10^{-8}$  s for a 30 cm diameter counter, which might be good enough for many purposes; oblongated forms are not advisable for this type.

## The Grain Density and the Process of Track Formation in Nuclear Emulsions (\*).

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(ricevuto il 10 Aprile 1954)

### PART III.

#### EXPERIMENTAL DETERMINATION OF THE PROBABILITY THAT A GRAIN BE RENDERED DEVELOPABLE

**Summary.** — After some considerations on the gap-length distribution, we suggest a method by which the probability that a grain be rendered developable can be obtained experimentally. Following our scheme on the process of track formation, we also indicate how it is possible to take into account the dip of the track. An example of application of our method to the mass identification of particles is given.

In two recent papers <sup>(1)</sup> the process by which an ionizing particle forms a track in a nuclear emulsion was analyzed in detail in the light of a new scheme.

According to this scheme, we introduced a parameter  $\pi$  (probability that a grain be rendered developable) which is, as the grain density, a function of  $dE/dR$  but, within large limits, is independent of the degree of development of the emulsion. We further demonstrated that, for a given track, the probability  $\pi$  may be deduced directly from the ratio between the total length

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<sup>(1)</sup> M. DELLA CORTE, M. RAMAT and L. RONCHI jr.: *Nuovo Cimento*, **10**, 509 (1953), Part I; **10**, 958 (1953), Part. II. We shall henceforth refer briefly to these two papers as (I) and (II).

(\*) This paper is part of a program of work supported by I.N.F.N., Sezione di Roma.

of the gaps,  $L_g$ , and their number,  $n_g$  (i.e., from the average length of a gap  $L_g/n_g$ ), as

$$(1) \quad \pi = \frac{n_g}{L_g} \cdot d_0,$$

where the constant  $d_0$  is the average length of the gaps corresponding to one undeveloped grain.

As a rough approximation, we identified  $d_0$  with the average diameter of the undeveloped grains, and attributed to it the numerical value of  $0.2 \mu$  given by G. BARONI and C. CASTAGNOLI for Ilford G5 emulsions <sup>(2)</sup>.

Experimental results have confirmed the validity of our basic assumptions, and shown that the evaluation of  $\pi$ , obtained from measurements of the average gap-length, is a very reliable datum for the identification of particles, and presents some notable advantages over those obtained by previous standard methods. We now propose to discuss, from a practical point of view, the experimental technique to be used for the determination of  $\pi$ , and the corrections to be applied in the case of tracks at an angle with the plane of the emulsion.

### 1. — The gap distribution function.

For a better understanding of what follows, we shall first briefly consider the gap-length distribution function.

According to our assumptions, the probability that the length of a gap be  $l$  is defined only if  $l$  is an integer multiple of  $d_0$  [(I) section 2].

Thus the gap-width distribution appears to be discontinuous, the successive terms being in a geometric progression. This is a logical consequence of having neglected: a) the fluctuations of the diameter of the undeveloped grains about the average value; b) the random distribution of their centers around the path of the particle; c) the fluctuations of the width of the gelatine layer separating neighbouring grains in the emulsion. From these and other minor causes it follows that the gap-width distribution becomes continuous.

Now, the most reasonable fit to our discontinuous distribution (geometric progression) is obviously given by an exponential function. This is in agree-

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(2) According to a more recent series of measurements (C. WALLER, private communication), the magnitude of the mean diameter appears to be  $0.3 \mu$ ; however, if we take into account the fact that the grain centers are distributed at random about the path of the particle, we get a better approximation to the truth by identifying  $d_0$  with the average value of the cord of the grains, assumed to be spherical. As this average value is  $2/3$  of the diameter, we again find  $d_0 = 0.2 \mu$ .



ment with the experimental results of O'CEALLAIGH<sup>(3)</sup>, who assumes this function to be given by

$$(2) \quad F(l)dl = \frac{A}{a} \exp \left[ -\frac{l}{a} \right] dl,$$

(for values of  $l$  that are not too small). The relation that must hold between the two constants  $A$  and  $a$  in formula (2) and our two parameters  $\pi$  and  $k$  (the latter characterizes the degree of development of the emulsion [(I) section 2]) is easily derived.

In our scheme, only gaps corresponding to at least one undeveloped grain (i.e., of length  $> d_0$ ) were considered. To take into account also gaps of length  $< d_0$  expression (8) of (I) should be summed over from 0 to  $\infty$ . The number of gaps in a given section of track corresponding to  $N$  grains of the undeveloped emulsion is then

$$n_g = \pi(1 - \pi)^{k-1}N.$$

By normalizing (2) over the total length of the track under observation, we derive from (9) (II) the expression

$$L_b = \frac{A}{a} \int_0^{\infty} \exp \left[ -\frac{l}{a} \right] l dl = d_0(1 - \pi)^k N$$

and, for the number of gaps

$$n_g = \frac{A}{a} \int_0^{\infty} \exp \left[ -\frac{l}{a} \right] dl = \pi(1 - \pi)^{k-1}N.$$

From the above two relations, we deduce

$$A = \pi(1 - \pi)^{k-1}N; \quad a = d_0 \frac{1 - \pi}{\pi}.$$

And hence follows the relation replacing relation (1)

$$(3) \quad \pi = \frac{d_0}{(L_b/n_g) + d_0}.$$

However, since the errors involved in gap measurements become increasingly significant as the gap-lengths decrease, it appears advisable to select « a priori »

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<sup>(3)</sup> C. O'CEALLAIGH: *C.R. du Congrès International sur le Rayonn. Cosm.*, Bagnères de Bigorre, 1953.

a minimum value,  $\varepsilon$ , for the gap-lengths to be measured, and to neglect all others. Now, assuming the gap length distribution to be exponential, if one neglects gaps whose length is  $< \varepsilon$ , the average value of the remaining ones  $(L_b/n_g)_{>\varepsilon}$  in related to the average value of the total number of gaps  $(L_b/n_g)$  by the expression (4)

$$\left(\frac{L_b}{n_g}\right)_{>\varepsilon} = \left(\frac{L_b}{n_g}\right) + \varepsilon.$$

Thus, by carrying out our measurements with a cut-off  $\varepsilon$ , we shall have

$$(4) \quad \pi = \frac{d_0}{(L_b/n_g)_{>\varepsilon} - \varepsilon + d_0}.$$

The fact that our first measurements reported in (I) and (II) were in good agreement with a scheme which did not take into account the continuity of the gap length distribution is not surprising, in view of the fact that the fluctuations which are responsible for the smoothing of the distribution, because of their random character, do not appreciably affect average values based on comparatively large numbers of gaps.

The discrepancy between formulae (1) and (4) is more apparent than real, because actually the finite resolving power of the microscope introduces an instrumental cut-off  $\varepsilon_0$ , so that the experimental values of the ratio  $L_b/n_g$  found by us were actually the values of  $(L_b/n_g)_{>\varepsilon_0}$ ; since  $\varepsilon_0$  has a value very near to  $d_0$ , (1) and (4) give the same values for  $\pi$ . In the above considerations, we have implicitly assumed the validity of (2) even for  $l < d_0$  when it surely ceases to hold.

We note, however, that the form of the distribution in this interval is immaterial. In fact, if we start by assuming a distribution of the form

$$F(l) dl = \begin{cases} 0 & 0 > l < d_0 \\ \frac{A}{a} \exp\left[-\frac{l}{a}\right] dl & l \geq d_0 \end{cases}$$

and proceed as above, we are again led to expression (3), and hence, by introducing the cut-off value  $\varepsilon$ , to relation (4).

## 2. - Experimental determination of $\pi$ .

As we have seen, the experimental determination of  $\pi$  involves only the measurement of the average gap-length,  $(L_b/n_g)_{>\varepsilon}$ . The choice of the limiting value  $\varepsilon$  depends on the visual aspect of the track under consideration:  $\varepsilon$  should

(4) C. O'CEALLAIGH: vedi nota (3).

be small for tracks that appear almost « black » and increase in magnitude with increasing lacunarity. From a practical standpoint, it is convenient to select for  $\varepsilon$  a value corresponding to an integral number of drum-divisions of the filar micrometer eye-piece. Measurements of gap-lengths can be readily

carried out by using a microscope whose stage has one of its micrometric movements sufficiently multiplied to allow an easy alignment of the gap edges with the hair of the drum-micrometer.

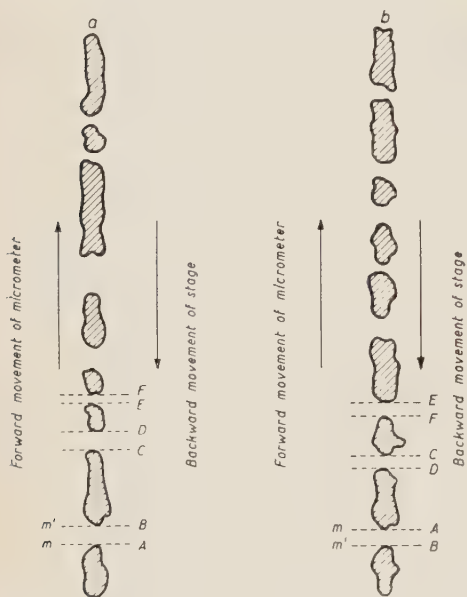


Fig. 1. — Gap-measuring technique: a) opposite movements of filar micrometer and stage: clot-lengths  $BC + DE + \dots = L_n$  are accumulated on screw-micrometer; gap-lengths  $AB + CD + \dots = L_b$  are accumulated on drum-micrometer; b) concordant movements of filar micrometer and stage: clot-lengths  $BC + DE + \dots = L_n + 2L_b$  are accumulated on screw-micrometer, gap-lengths  $AB + CD + \dots = L_b$  are accumulated on drum-micrometer.

In this Laboratory a Koristka microscope has been used, equipped with a rotating stage, whose forward movement is controlled by a micrometer screw of 0.5 mm pitch. The plate is set so that the section of track to be observed is parallel to the forward movement; the hair of the micrometer is then set tangent to the first edge of the gap (position A, Fig. 1a). If now we advance the hair until it is tangent to the second edge (position B) and subsequently move the track forward by means of the micrometer screw along the black section BC, and so on, we shall sum up the total length of the gaps ( $L_b$ ) on the drum micrometer, and the total length of the grain-groups ( $L_n$ ) on the screw-micrometer. When the hair has swept over the whole field of the microscope, one may move it backwards, following the same technique (to note, that in this case, one adds up on the screw-micrometer the sum  $L_n + 2L_b$  (Fig. 1-b). The gaps to be measured are evidently only those whose length exceeds the selected value for  $\varepsilon$ . In most cases, it is possible to judge at first sight whether a gap is to be considered or ignored; when any doubt arises, the observer notes the positions  $m$  and  $m'$  of the micrometer corresponding to the edges of the gap; should  $m - m'$  turn out to be  $< \varepsilon$ , he moves back the hair to position  $m$ , and proceeds as if the gap were non existent.



To count the gaps while measuring their length, we use a recorder driven by a key. This simple device helps the observer to better concentrate his attention on the alignment of the hair, as the counting becomes semi-automatic. By means of the above technique, the determination of  $\pi$  on  $100 \div 200$  gaps (which in most cases are sufficient) requires about the time of a normal measurement of scattering (the subsequent numerical evaluation of  $\pi$  involving only very easy and speedy calculations).

We may add, that values obtained for the same track by different observers, after a very short training, are consistent to within a  $1 \div 2\%$ .

### 3. - Corrections for tracks at an angle with the plane of the emulsion.

If a track is steeply dipped, the apparent length of the gaps is less than the true one, and a certain number of small gaps becomes unobservable. To take this fact into due account, let us consider a gap in the unshrunk emulsion (Fig. 2). We shall assume that the shrinkage occurring during the drying stage does not alter the projected length of the gaps. Let  $l$  be the actual length, and  $A$  and  $B$  the positions of the two nearest developable grains; then, in accordance with the assumptions we made about the swelling of the grains during the developing process, we have

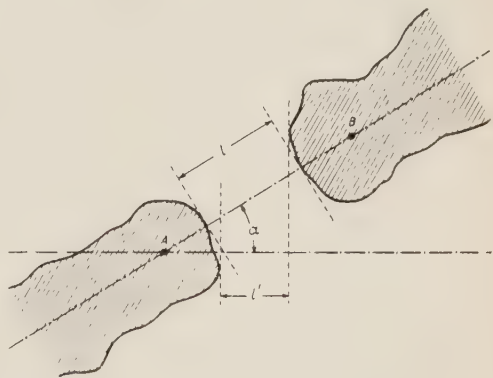


Fig. 2. - Dip-correction:  $l$  actual gap-length;  $l'$  apparent gap-length;  $\alpha$  angle of dip on the unshrunk emulsion.

$$l = \frac{l'}{\cos \alpha} + (k-1)d_0 \left( \frac{1}{\cos \alpha} - 1 \right),$$

where  $\alpha$  is the angle formed by the track with the plane of the emulsion. The same relation will hold both for the average true gap-length and the true magnitude of  $\varepsilon$ , in terms of the average apparent gap-length and of the apparent magnitude  $\varepsilon'$  of  $\varepsilon$ , respectively. Therefore

$$\left( \frac{L_b}{n_g} \right)_{>\varepsilon} - \varepsilon = \frac{\left( \frac{L_b}{n_g} \right)' - \varepsilon'}{\cos \alpha}.$$

Gaps which become invisible on account of the dip, need not be taken into account, as their length is certainly less than the selected cut-off. For tracks

that show a marked angle of dip, one has therefore

$$(5) \quad \pi = \frac{d_0}{(L_b/n_d) > \varepsilon - \varepsilon} \cdot \frac{\varepsilon}{\cos \alpha} + d_0$$

The above correction becomes obviously useless only if the dip is so very pronounced that the number of visible gaps results exceedingly small, as measurements then lose statistical significance. The possibility of considering also strongly dipped tracks is one of the advantages of our method as compared to others; in fact, in all methods based on a measurements of only the number, or only the length of gaps, or on normal grain counting, corrections for the effect of dip can be introduced only in case of very small angles.

#### 4. - Preliminary results in the application of the method to mass determination.

We have applied our method to the determination of the mass of particles due to the interaction of  $\pi^\pm$  artificial mesons of  $\sim 70$  MeV. The measurements given here are only the preliminary results of a systematic study of the energy spectrum of such particles. The plates were Ilford G.5 of 400 and 600  $\mu$ , processed according to standard techniques.

Of every track whose length exceeded 1000  $\mu$  and did not end in the emulsion, we measured the average angle of scattering  $\bar{\alpha}$  ( $^\circ/100 \mu$ ), and the value of  $\pi$  over a few hundred microns. The microscope used was a Koristka, with a  $\times 100$  objective and a filar drum-micrometric eye-piece. The reticle consisted of a very thin spider's thread: the drum was divided into 100 divisions. Selected values for  $\varepsilon$  were 0.208  $\mu$  for « black » tracks, and 0.416  $\mu$  for « gray » tracks (10 and 20 drum divisions, respectively). Results are given in Fig. 3. The full lines were derived from the relation between  $\pi$  and energy loss (II) and from the function  $dE/dR = f(R)$  deduced from the range energy curves given by VIGNERON <sup>(5)</sup>.

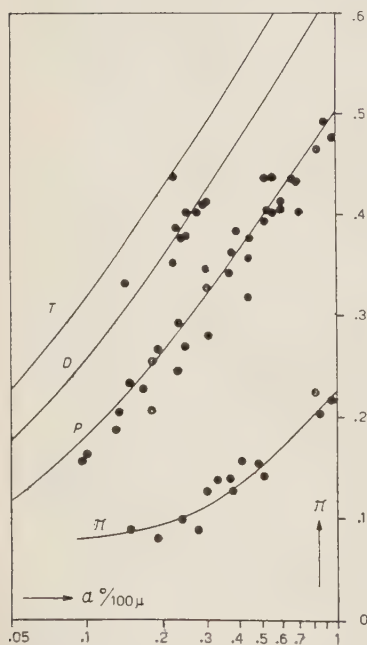


Fig. 3. - Results of the measurements of  $\pi$  and  $\bar{\alpha}$  ( $^\circ/100 \mu$ ).

(5) L. VIGNERON: *Journ. de Phys. et Rad.*, **14**, 145 (1953).

The mass spectrum which was obtained from  $\sim 60$  tracks is shown in Fig. 4. In evaluating the above results, one must bear in mind that the tracks were observed in two plates which in some sectors presented differences in the degree of development recognizable at first sight.

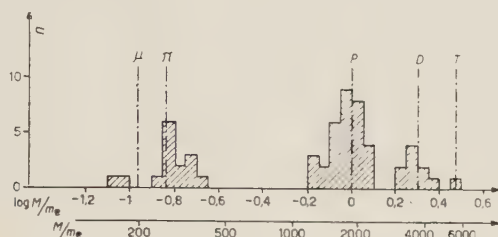


Fig. 4. — Mass spectrum obtained from the data of Fig. 3.

As many of the tracks were no longer than  $1500 \div 2000 \mu$ , we have good reason to assume, also in view of other considerations, that the dispersion of the experimental points is to be attributed to the uncertainty of the scattering measurements, rather than to the determination of  $\pi$ .

Apart from the meson tracks, and a few of the proton tracks, the mass identification would have been practically impossible by standard methods of scattering-grain density measurements, not only on account of the high value of grain density, but also because such measurements would have been notably affected by the above mentioned differences in the degree of development.

Concluding, we think it legitimate to assume that the experimental evaluation of  $\pi$  by means of the average gap-length offers some very notable advantages in view of the identification of particles, as compared to grain and blob counting, or to measurements which involve solely number and length of tracks: in point of fact, it is independent from the degree of development, and may be easily corrected so as to take into account the effect of dip.

We wish to express our thanks to Professor S. FRANCHETTI for his continued interest in this work, and to Professor M. MANDÒ for many helpful discussions. Thanks are also due to Ing. GEBENDINGER for his valuable help in collecting experimental data. We are indebted to Mr. C. WALLER, of the Ilford Co. Ltd. for the useful information regarding the mean diameter of undeveloped grains.



## RIASSUNTO

Considerando una funzione di distribuzione esponenziale per le lunghezze delle lacune, è possibile ottenere una espressione della probabilità  $\pi$  di impressionamento di un granulo in cui interviene la lunghezza media delle lacune maggiori di un limite prefissato. Le piccolissime lacune, che contribuiscono notevolmente all'imprecisione del risultato, possono quindi essere trascurate nella misura. Sulla base del nostro schema per il processo di formazione della traccia viene indicato il modo di correggere i risultati di misure di  $\pi$  per tracce inclinate sul piano dell'emulsione. Come esempio di applicazione delle misure di  $\pi$  si presentano i risultati ottenuti nell'identificazione di tracce prodotte nelle interazioni di mesoni  $\pi^\pm$  artificiali.

## Guidage rectiligne sur lames élastiques. Comparaison de divers types connus et nouveaux.

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(ricevuto il 17 Aprile 1954)

**Résumé.** — Le présent travail développe une méthode générale de calcul des suspensions élastiques simples et à compensation, où les lames sont flexibles sur toute leur hauteur. Il débute par le cas d'une lame élémentaire étudiée par le développement en série de Fourier de sa déformée. La méthode permet, moyennant des hypothèses très générales, de déterminer le raccourcissement vertical des lames. Les suspensions compensées, tant asservies que libres, sont étudiées et comparées entre elles. En particulier on met en évidence les avantages de la suspension asservie. Le cas où les suspensions élémentaires sont de hauteurs différentes est traité. Plusieurs suspensions nouvelles réalisant une trajectoire rectiligne même si la platine est pesante, sont proposées. Les modifications sont simples et facilement calculables. Leur ajustement s'effectue indépendamment des autres réglages de la suspension.

Dans un précédent travail <sup>(1)</sup> nous avons développé une méthode générale de calcul du mouvement parasite de suspensions symétriques tant simples que compensées. Cette méthode conduit malheureusement à des expressions peu maniables mettant en jeu des intégrales elliptiques. Aussi nous a-t-il paru intéressant de développer une méthode approchée n'impliquant cependant que des hypothèses très générales. C'est l'objet du présent travail.

La méthode de calcul que nous développons s'applique aux suspensions simples en U et en U renversé ainsi qu'à leur combinaison: la suspension compensée. Celle-ci peut appartenir au type libre ou non asservi comme nous

<sup>(1)</sup> J. E. PLAINEVAUX: *Nuovo Cimento*, **10**, 1451 (1953).

l'avons schématisé à la fig. 1 où les lames ont été dessinées alternées suivant une idée de CLAY. Si on permute les deux lames de gauche du dessin on réalise une suspension en sandwich (suspension de platine encadrée par la suspension du cadre raidisseur). En ce qui concerne les propriétés que nous étudions ici ces deux suspensions sont équivalentes.

La suspension peut aussi être du type asservi que nous avons schématisé à la fig. 2. Ici encore la disposition des lames est indifférente. Dans la fig. 2

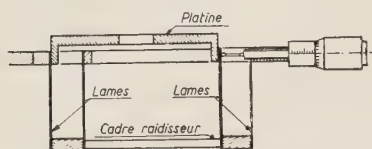


Fig. 1.

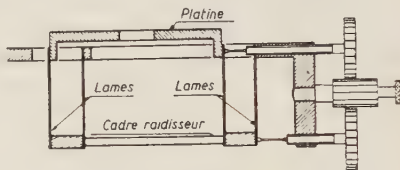


Fig. 2.

nous avons supposé que l'asservissement réalisait le rapport de deux à un des déplacements par deux vis tournant au synchronisme dont les pas sont dans le rapport de deux à un. Bien d'autres solutions sont possibles, auxquelles la méthode de calcul présentée s'applique encore. En particulier la méthode est encore valable si le rapport des déplacements est différent de deux à un, généralisant ainsi ce type de suspension. On montrera que cette généralisation conduit à un cas particulier fort intéressant.

Au cours de cette étude nous retrouverons comme cas particulier une partie des résultats trouvés par nous dans un article précédent <sup>(2)</sup>. Il nous sera possible de préciser le domaine d'application des formules trouvées dans cette étude précédente.

La méthode de calcul, utilisant le développement en série de Fourier de la courbe déformée des lames permet une comparaison entre les suspensions compensées libres et asservies, qui conduit à réclamer pour la suspension asservie des avantages indiscutables. Enfin la méthode de calcul nous a permis de trouver plusieurs types de suspension réalisant une trajectoire rigoureusement rectiligne au degré d'approximation introduit par les hypothèses de calcul qui sont très générales, surtout si les déformations sont peu importantes. Ces suspensions, que nous croyons nouvelles, sont obtenues par de légères modifications des suspensions connues et ne sont pas plus coûteuses; elles consistent en l'utilisation de lames de moments d'inertie différents, ou de longueurs différentes, ou d'asservissement de rapport particulier différent du rapport simple de deux à un. Le réglage fin de la trajectoire parasite dans le

(<sup>2</sup>) J. E. PLAINEVAUX: *Nuovo Cimento*, 11, 626 (1954).



plan vertical n'introduit aucune sujétion particulière et n'interfère pas avec les autres réglages de la suspension.

Remarquons que les modifications que nous avons découvertes ainsi par le calcul sont maintenant faciles à trouver, tout au moins qualitativement, par un raisonnement simple ne faisant intervenir que des considérations très élémentaires d'élasticité.

On peut se demander si le mouvement parasite dans le plan vertical mérite toute l'attention que nous lui portons. Sans doute un faible mouvement parasite n'est-il souvent *en soi* qu'un défaut mineur de l'appareil. Mais ce mouvement parasite a plusieurs conséquences graves dont nous ne citerons que la nécessité où se trouve le constructeur d'utiliser des lames fort rigides pour que le mouvement ne soit pas gênant. Cette forte rigidité a des conséquences particulièrement néfastes pour la vis et pour la platine à laquelle la vis va transmettre des forces et couples parasites d'autant plus importants que les forces mises en jeu sont grandes. A ce propos, tout dispositif permettant de diminuer les forces ainsi transmises est très intéressant. Une possibilité de maintenir la force horizontale constante est à souhaiter.

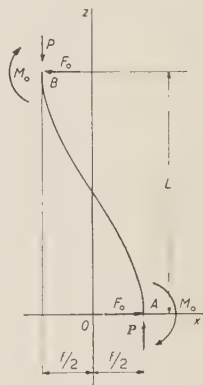


Fig. 3.

### Étude de la déformation d'une lame de suspension.

Considérons (fig. 3), une lame  $AB$  de longueur  $L$  sollicitée par des forces horizontales  $F_0$  et verticales  $P$ , ainsi que par des moments d'extrémités réalisant la verticalité des tangentes aux limites  $A$  et  $B$ . Développons l'équation de la déformée en série de Fourier, chaque terme vérifiant les conditions d'encaastrements aux limites. La fonction générale  $\cos n\pi z/L$  satisfait à ces conditions.

La déformée aura donc pour développement :

$$x = \sum_1^{\infty} a_n \cos \frac{n\pi z}{L},$$

dont nous devons calculer les coefficients  $a_n$ .

Au point  $A$ , la demi-flèche  $f/2$  de la lame vaut :

$$\frac{f}{2} = x_{z=0} = \sum_1^{\infty} a_n,$$

tandis qu'au point  $B$  la demi-flèche  $f/2$  vaudra :

$$\frac{f}{2} = - \sum_1^{\infty} a_n \cos n\pi.$$

D'autre part, du fait de la déformation de la lame il y a raccourcissement de celle-ci suivant la verticale, le point  $B$  descend d'une certaine longueur  $\lambda$  donnée sensiblement par

$$\lambda = \frac{1}{2} \int_0^L \left( \frac{dx}{dz} \right)^2 dz.$$

Or,

$$\frac{dx}{dz} = -\frac{\pi}{L} \sum_1^{\infty} a_n \sin \frac{n\pi z}{L}.$$

Elevant au carré et intégrant il vient:

$$\lambda = \frac{\pi^2}{4L} \sum_1^{\infty} n^2 a_n^2.$$

Calculons encore l'énergie de déformation de flexion de la lame, elle vaut:

$$U = \frac{EI}{2} \int_0^L \left( \frac{d^2x}{dz^2} \right)^2 dz.$$

$I$  étant le moment d'inertie de la lame par rapport à sa fibre neutre et  $E$  le module d'élasticité du matériau constituant la lame. Si besoin est, on remplacera dans cette formule ainsi que dans toutes les suivantes,  $E$  par le module réduit  $E' = E(1 - \nu^2)$ ,  $\nu$  étant le coefficient de Poisson.

Ayant:

$$\frac{d^2x}{dz^2} = -\frac{\pi^2}{L^2} \sum_1^{\infty} n^2 a_n \cos \frac{n\pi z}{L},$$

élevant au carré et intégrant, on obtient:

$$U = \frac{EI\pi^4}{4L^3} \sum_1^{\infty} n^4 a_n^2.$$

On sait que lorsqu'un système élastique est soumis à un petit déplacement à partir de sa position d'équilibre, l'accroissement correspondant de l'énergie potentielle du dit système est égal au travail effectué par les forces extérieures durant le déplacement considéré. Lorsque la déformée est donnée par la série:

$$x = \sum_1^{\infty} a_n \cos \frac{n\pi z}{L},$$

on obtient les petits déplacements autour de la position d'équilibre par de faibles variations des coefficients  $a_n$ .

Si l'on donne un accroissement  $da_n$  au coefficient  $a_n$ , on a dans la série précédente

$$(a_n + da_n) \cos \frac{n\pi z}{L},$$

au lieu du terme

$$a_n \cos \frac{n\pi z}{L},$$

tandis que les autres termes restent inchangés.

L'accroissement  $da_n$  représente une petite déformation additionnelle donnée par la cosinussoïde

$$da_n \cos \frac{n\pi z}{L},$$

superposée à la déformation d'équilibre. Pendant cette déformation additionnelle les forces extérieures effectuent un certain travail que nous allons calculer. Ce travail se compose d'abord du travail de la force  $F_0$  appliquée au point  $A$  et qui vaut :

$$F_0 da_n,$$

ensuite du travail de la force  $F_0$  appliquée en  $B$ , dont l'expression est :

$$- F_0 \cos n\pi \cdot da_n.$$

Soit au total :

$$F_0(1 - \cos n\pi) da_n.$$

D'autre part le point  $B$  descend de

$$d\lambda = \frac{\partial \lambda}{\partial a_n} da_n = \frac{\pi^2}{2L} n^2 a_n da_n,$$

donnant lieu pour  $P$  au travail

$$P d\lambda = \frac{P\pi^2}{2L} n^2 a_n da_n.$$

Les couples d'encastrements ne travaillent pas lors de la déformation virtuelle utilisée.



Considérons maintenant l'accroissement de l'énergie de déformation résultant de l'accroissement  $da_n$  de  $a_n$ . Il vaut :

$$dU = \frac{\partial U}{\partial a_n} da_n = \frac{EI\pi^4}{2L^3} n^4 a_n da_n.$$

Egalant cette expression à la somme des travaux effectués par les forces extérieures, il vient :

$$\frac{EI\pi^4}{2L^3} n^4 a_n = F_0(1 - \cos n\pi) + \frac{P\pi^2}{2L} n^2 a_n,$$

d'où :

$$a_n = \frac{F_0(1 - \cos n\pi)}{\frac{EI\pi^4}{2L^3} n^2 \left( n^2 - \frac{PL^2}{EI\pi^2} \right)},$$

ce qui détermine les coefficients de la série et permet d'écrire l'équation de la déformée sous la forme :

$$x = \frac{2F_0L^3}{EI\pi^4} \sum_1^{\infty} \frac{1 - \cos n\pi}{n^2 \left( n^2 - \frac{PL^2}{EI\pi^2} \right)} \cos \frac{n\pi z}{L}.$$

Cette fonction ne présente, outre le fondamental, que des harmoniques impairs.

Nous pouvons simplifier cette formule en introduisant une variable réduite. Pour cela, imaginons une lame, analogue à celle que nous étudions, sollicitée uniquement par la force verticale  $P$  et des moments d'encastrement maintenant les tangentes verticales aux extrémités.

Une telle lame aura, vis-à-vis du flambage, la charge critique d'Euler :

$$P_c = \frac{\pi^2 EI}{L^2}.$$

Introduisons le rapport de l'effort vertical  $P$  appliqué à la lame à sa valeur critique  $P_c$ . Ce rapport :

$$\alpha = \frac{P}{P_c} = \frac{PL^2}{\pi^2 EI},$$

est ce que nous désignerons sous le nom de charge réduite ou « paramètre » de la lame élastique. Nous ferons grand usage de cette notion. Dans les cas

simples, le paramètre  $\alpha$  s'identifie à l'inverse du coefficient de sécurité sur le flambage.

Avec la charge réduite  $\alpha$  le coefficient général de la série de Fourier représentant la déformée s'écrit :

$$a_n = \frac{2F_0 L^3}{\pi^4 EI} \cdot \frac{1 - \cos n\pi}{n^2(n^2 - \alpha)},$$

et la déformée s'écrit :

$$x = \frac{2F_0 L^3}{\pi^4 EI} \sum_1^{\infty} \frac{1 - \cos n\pi}{n^2(n^2 - \alpha)} \cos \frac{n\pi z}{L}.$$

**Calcul de la flèche horizontale  $f$ .**

Nous avons trouvé précédemment que l'on a ;

$$f = 2 \sum_1^{\infty} a_n;$$

introduisant la valeur trouvée pour  $a_n$ , on obtient

$$f = \frac{4F_0 L^3}{\pi^4 EI} \sum_1^{\infty} \frac{1 - \cos n\pi}{n^2(n^2 - \alpha)}.$$

Le terme de sommation pouvant encore s'écrire

$$\frac{1}{\alpha} \sum_1^{\infty} \frac{1 - \cos n\pi}{n^2 - \alpha} - \frac{1}{\alpha} \sum_1^{\infty} \frac{1 - \cos n\pi}{n^2},$$

or, on a :

$$\sum_1^{\infty} \frac{1 - \cos n\pi}{n^2 - \alpha} = \frac{\pi}{2\sqrt{\alpha}} \frac{1 - \cos \pi\sqrt{\alpha}}{\sin \pi\sqrt{\alpha}} \quad \text{et} \quad \sum_1^{\infty} \frac{1 - \cos n\pi}{n^2} = \frac{\pi^2}{4},$$

ce qui permet d'écrire l'expression de la flèche sous la forme :

$$f = \frac{4F_0 L^3}{\pi^4 EI \alpha} \left[ \frac{\pi}{2\sqrt{\alpha}} \frac{1 - \cos \pi\sqrt{\alpha}}{\sin \pi\sqrt{\alpha}} - \frac{\pi^2}{4} \right],$$

que nous écrirons encore :

$$f = \frac{F_0 L^3}{12 EI} \Phi(\alpha),$$

en posant

$$\Phi(\alpha) = \frac{3 \left( \operatorname{tg} \frac{\pi}{2} \sqrt{\alpha} - \frac{\pi}{2} \sqrt{\alpha} \right)}{\left( \frac{\pi}{2} \sqrt{\alpha} \right)^3}.$$

La fonction  $\Phi$  étant toujours supérieure ou égale à l'unité, valant l'unité pour la valeur nulle du paramètre et tendant vers l'infini quand le paramètre  $\alpha$  tend vers l'unité. On trouvera (fig. 4), la courbe de  $\Phi$  en fonction de  $\alpha$ .

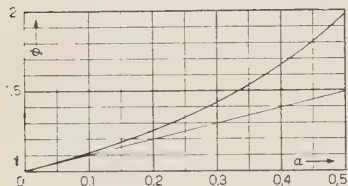


Fig. 4.

Des tables de cette fonction se trouvent dans la *Théorie de la Stabilité élastique* de S. TIMOSHENKO. Si l'on désire des valeurs plus précises on s'aidera des tables de la fonction  $\operatorname{tg} x - \operatorname{arc} x$  qui est bien connue des calculateurs d'engrenages en développante de cercle. C'est en effet la fonction  $\operatorname{inv} x$  (funzione dell'evolvente). Les tables les plus

complètes sont dûes à EARLE BUCKINGHAM <sup>(3)</sup>.

Enfin, pour de petites valeurs de  $\alpha$ , on utilisera le développement en série suivant :

$$\Phi(\alpha) = 1 + \frac{\pi^2}{10} \alpha + \frac{17\pi^4}{1680} \alpha^2 + \frac{31\pi^6}{30240} \alpha^3 + \dots$$

Remarquons que les séries que nous utilisons convergent fort rapidement. Ainsi pour l'expression de la flèche

$$f = \frac{4F_0 L^3}{\pi^4 EI} \sum_1^{\infty} \frac{1 - \cos n\pi}{n^2(n^2 - \alpha)},$$

bornons-nous à prendre un seul terme, nous obtenons dans le cas particulier  $\alpha = 0$

$$f \approx \frac{F_0 L^3}{(\pi^4/8)EI} = \frac{F_0 L^3}{12,18 EI},$$

alors que la formule exacte est :

$$\frac{F_0 L^3}{12 EI},$$

soit un erreur de 1,5%.

<sup>(3)</sup> EARLE BUCKINGHAM: *Manual of gear design*, section one (New York, 1935).



Malheureusement cette simplification, que l'on peut faire ici, n'est plus admissible dans la suite des calculs. C'est pourquoi nous nous sommes imposé de sommer toutes les séries que nous rencontrerons sans introduire de simplification.

### Calcul du raccourcissement vertical $\lambda$ .

Nous avons montré que l'abaissement du point  $B$  ou raccourcissement vertical  $\lambda$  était sensiblement donné par

$$\lambda = \frac{1}{2} \int_0^L \left( \frac{dx}{dz} \right)^2 dz = \frac{\pi^2}{4L} \sum_1^{\infty} n^2 a_n^2.$$

Introduisant la valeur trouvée pour  $a_n$ , on obtient:

$$\lambda = \left( \frac{F_0}{EI} \right)^2 \frac{L^5}{\pi^6} \sum_1^{\infty} \frac{(1 - \cos n\pi)^2}{n^2(n^2 - \alpha)^2},$$

le terme de sommation pouvant se remplacer par:

$$\frac{2}{\alpha} \sum_1^{\infty} (1 - \cos n\pi) \left[ \frac{1}{(n^2 - \alpha)^2} - \frac{1}{\alpha} \cdot \frac{1}{n^2 - \alpha} + \frac{1}{\alpha} \cdot \frac{1}{n^2} \right].$$

On trouve que:

$$\begin{aligned} \sum_1^{\infty} \frac{1 - \cos n\pi}{(n^2 - \alpha)^2} &= \frac{\pi^2}{4\alpha} \frac{1 - \cos \pi\sqrt{\alpha}}{\sin^2 \pi\sqrt{\alpha}} - \frac{\pi}{4\alpha\sqrt{\alpha}} \cdot \frac{1 - \cos \pi\sqrt{\alpha}}{\sin \pi\sqrt{\alpha}}, \\ \sum_1^{\infty} \frac{1 - \cos n\pi}{n^2} &= \frac{\pi^2}{4}, \quad \sum_1^{\infty} \frac{1 - \cos n\pi}{n^2 - \alpha} = \frac{\pi}{2\sqrt{\alpha}} \cdot \frac{1 - \cos \pi\sqrt{\alpha}}{\sin \pi\sqrt{\alpha}}. \end{aligned}$$

Remplaçant les séries par leurs sommes, on obtient:

$$\lambda = \frac{L^5}{240} \left( \frac{F_0}{EI} \right)^2 \left[ \frac{15}{4} \frac{\frac{\pi}{2} \sqrt{\alpha} \left( 3 + \operatorname{tg}^2 \frac{\pi}{2} \sqrt{\alpha} \right) - 3 \operatorname{tg} \frac{\pi}{2} \sqrt{\alpha}}{\left( \frac{\pi}{2} \sqrt{\alpha} \right)^5} \right],$$

que nous écrirons

$$\lambda = \frac{L^5}{240} \left( \frac{F_0}{EI} \right)^2 \Psi(\alpha),$$

$\Psi(\alpha)$  représentant la fonction comprise entre les crochets.

On trouvera (fig. 5) un graphique de cette fonction. Son minimum vaut l'unité pour la valeur zéro du paramètre; elle tend vers l'infini quand le paramètre se rapproche de l'unité.

Pour de petites valeurs du paramètre  $\alpha$  on utilisera le développement en série de puissances suivant:

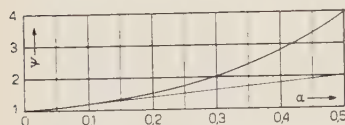


Fig. 5.

$$\Psi(\alpha) = 1 + \frac{17\pi^2}{84}\alpha + \frac{31\pi^4}{1008}\alpha^2 + \frac{691\pi^6}{166\,320}\alpha^3 + \dots$$

A côté de la fonction  $\Psi$  qui permet de calculer le raccourcissement vertical en fonction de la force horizontale  $F_0$ , il y a aussi lieu de rechercher une fonction permettant de calculer le raccourcissement vertical  $\lambda$  lorsque la flèche horizontale  $f$  est donnée.

#### Raccourcissement vertical $\lambda$ en fonction de la flèche horizontale $f$ .

Sachant que la flèche horizontale  $f$  est donnée par:

$$f = \frac{F_0}{EI} \frac{L^3}{12} \Phi(\alpha),$$

et que le raccourcissement vertical  $\lambda$  a pour expression

$$\lambda = \frac{L^5}{240} \left( \frac{F_0}{EI} \right)^2 \Psi(\alpha),$$

l'élimination de  $F_0$  entre ces deux relations donne l'expression du raccourcissement vertical  $\lambda$  en fonction de la flèche horizontale  $f$ :

$$\lambda = \frac{3}{5} \frac{f^2}{L} \frac{\Psi(\alpha)}{\Phi^2(\alpha)} = \frac{3}{5} \frac{f^2}{L} \chi(\alpha),$$

avec:

$$\chi(\alpha) = \frac{\Psi}{\Phi^2} = \frac{5}{12} \frac{\pi}{2} \sqrt{\alpha} \frac{\frac{\pi}{2} \sqrt{\alpha} \left( 3 + \operatorname{tg}^2 \frac{\pi}{2} \sqrt{\alpha} \right) - 3 \operatorname{tg} \frac{\pi}{2} \sqrt{\alpha}}{\left( \operatorname{tg} \frac{\pi}{2} \sqrt{\alpha} - \frac{\pi}{2} \sqrt{\alpha} \right)^2}.$$

Les graphiques (fig. 6 et 7), donnent la courbe  $\chi(\alpha)$  en fonction du paramètre  $\alpha$ .

La fonction  $\chi$  est toujours supérieure ou égale à l'unité. Elle atteint cette

quantité pour la valeur zéro du paramètre  $\alpha$ . La fonction devient infinie lorsque :

$$\operatorname{tg} \frac{\pi}{2} \sqrt{\alpha} = \frac{\pi}{2} \sqrt{\alpha},$$

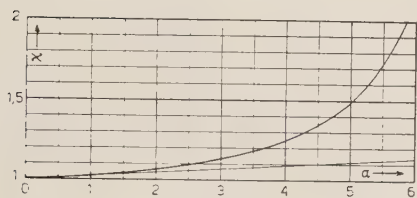


Fig. 6.

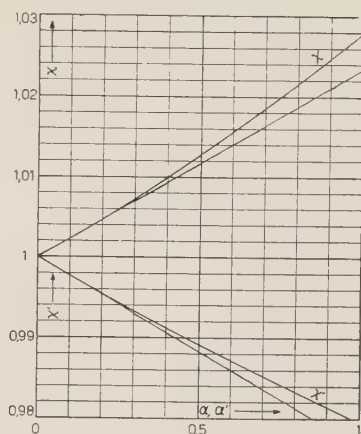


Fig. 7.

dont la plus petite racine positive non nulle est <sup>(4)</sup> :

$$\frac{\pi}{2} \sqrt{\alpha} = 4,4934,$$

ou :

$$\alpha = 8,1830.$$

Pour de petites valeurs du paramètre  $\alpha$  on utilisera le développement en série suivant :

$$\chi(\alpha) = 1 + \frac{\pi^2}{420} \alpha + \frac{\pi^4}{25200} \alpha^2 + \frac{37\pi^6}{58212000} \alpha^3 + \dots$$

### Moment fléchissant aux extrémités.

Calculons d'abord ce moment lorsque la force horizontale  $F_0$  est connue. Le moment fléchissant à l'extrémité  $A$  vaut :

$$M = \frac{\pi^2 EI}{L^2} \sum_1^\infty n^2 a_n = \frac{2F_0 L}{\pi^2} \sum_1^\infty \frac{1 - \cos n\pi}{n^2 - \alpha},$$

or, on a :

$$\sum_1^\infty \frac{1 - \cos n\pi}{n^2 - \alpha} = \frac{\pi}{2\sqrt{\alpha}} \cdot \frac{1 - \cos \pi\sqrt{\alpha}}{\sin \pi\sqrt{\alpha}},$$

<sup>(4)</sup> E. LOMMEL: *Abh. Münch. Akad.*, (2) 15, 123 (1886).



ce qui permet d'écrire le moment aux extrémités sous la forme:

$$M = \frac{F_0 L}{2} \frac{\operatorname{tg} \frac{\pi}{2} \sqrt{x}}{\frac{\pi}{2} \sqrt{x}},$$

que nous écrirons encore:

$$M = \frac{F_0 L}{2} \tau(x).$$

On trouvera à la fig. 8 un graphique de la fonction  $\tau$ . Remarquons que l'on a:

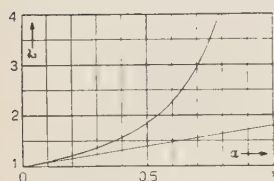


Fig. 8.

$$\tau(0) = 1,$$

$$\tau(1) = \infty,$$

la fonction étant toujours croissante.

Des tables de cette fonction se trouvent sans difficulté. Citons les tables classiques de JAHNKE-EMDE <sup>(5)</sup>.

Pour les petites valeurs du paramètre  $x$  on utilisera le développement en série suivant:

$$\tau(x) = 1 + \frac{\pi^2}{12} x + \frac{\pi^4}{120} x^2 + \frac{17\pi^6}{20160} x^3 + \dots$$

Le calcul du moment fléchissant régnant à l'extrémité A lorsque la flèche horizontale  $f$  est donnée se fait sans difficulté en éliminant la force  $F_0$  entre l'expression précédente de  $M$  et l'expression de  $f$ :

$$f = \frac{F_0}{EI} \frac{L^3}{12} \Phi(x).$$

On trouve:

$$M = \frac{6EI f}{L^2} \frac{\tau(x)}{\Phi(x)},$$

avec:

$$\frac{\tau(x)}{\Phi(x)} = \left( \frac{\pi}{2} \sqrt{x} \right)^2 \frac{\operatorname{tg} \frac{\pi}{2} \sqrt{x}}{3 \left( \operatorname{tg} \frac{\pi}{2} \sqrt{x} - \frac{\pi}{2} \sqrt{x} \right)}.$$

<sup>(5)</sup> E. JAHNKE, F. EMDE: *Tables of functions* (Dover, 1945).

La fig. 9 montre l'allure générale de cette fonction.

La fonction  $\tau/\Phi$  est reliée à la fonction «  $c$  » de Bleich <sup>(6)</sup> par la relation:

$$\frac{\tau}{\Phi} = \frac{1}{3c}.$$

Pour de petites valeurs du paramètre  $\alpha$  on se contentera d'utiliser le développement en série:

$$\frac{\tau}{\Phi} = 1 - \frac{\pi^2}{60} \alpha - \frac{\pi^4}{8\,400} \alpha^2 - \frac{\pi^6}{756\,000} \alpha^3 - \dots$$

Les formules que nous venons de développer permettent de calculer rapidement tous les éléments d'une suspension symétrique du type en U renversé où les lames sont sollicitées au flambage. N'oublions pas que les charges axées que l'on utilise en pratique sont toujours loin de la charge critique de flambage.

Suivant le cas envisagé, c'est-à-dire suivant que c'est la force horizontale qui est donnée ou la flèche horizontale; les charges critiques de flambage varient beaucoup, tous les autres éléments restant égaux. De ce fait, la valeur du paramètre  $\alpha$  à utiliser en pratique variera aussi dans d'assez fortes limites suivant le type de suspension. Cette remarque se montrera utile dans les suspensions du type à compensation, qui se composent d'une suspension en U renversé combinée à une suspension en U.

Afin d'être à même de calculer des suspensions compensées, il nous faut d'abord étudier les suspensions en U. Cette étude se fait d'une manière analogue à celle des suspensions en U renversé. Il n'y a plus ici danger de flambage, la charge verticale ayant un effet stabilisateur sur la lame. Cet effort tend en effet à redresser la lame.

Pour étudier ces lames soumises à traction, nous nous bornerons à dire que les calculs précédents restent valables à condition d'y changer le signe du paramètre  $\alpha$ . Le paramètre d'une lame chargée au redressement sera désigné par

$$\alpha' = \frac{P' L'^2}{\pi^2 E' I'},$$

nous affecterons toutes les valeurs de l'indice ' pour rappeler que la lame appartient à une suspension en U.

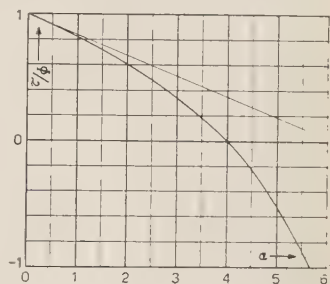


Fig. 9.

(6) F. BLEICH: *Buckling strength of metal structures* (New York, 1952), pp. 204 à 207.

On trouve alors immédiatement les formules suivantes:

*Flèche horizontale  $f'$ :*

$$f' = \frac{F'_0 L'^3}{12 E' I'} \frac{3 \left( \frac{\pi}{2} \sqrt{\alpha'} - \operatorname{th} \frac{\pi}{2} \sqrt{\alpha'} \right)}{\left( \frac{\pi}{2} \sqrt{\alpha'} \right)^3} = \frac{F'_0 L'^3}{12 E' I'} \Phi'(\alpha').$$

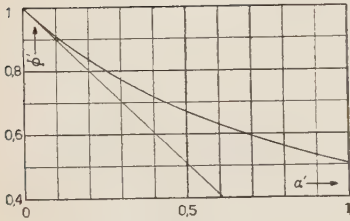


Fig. 10.

La fonction  $\Phi'$  (fig. 10) étant une fonction décroissante valant l'unité pour  $\alpha' = 0$  et tendant vers zéro lorsque le paramètre tend vers l'infini. Développement pour de petites valeurs de  $\alpha'$ :

$$\Phi' = 1 - \frac{\pi^2}{10} \alpha' + \frac{17\pi^4}{1680} \alpha'^2 - \frac{31\pi^6}{30240} \alpha'^3 + \dots$$

*Raccourcissement vertical  $\lambda'$ :*

$$\lambda' = \frac{L'^5}{240} \left( \frac{F'_0}{E' I'} \right)^2 \Psi'(\alpha'),$$

avec

$$\begin{aligned} \Psi'(\alpha') &= \\ &= \frac{15}{4} \frac{\frac{\pi}{2} \sqrt{\alpha'} \left( 3 - \operatorname{th}^2 \frac{\pi}{2} \sqrt{\alpha'} \right) - 3 \operatorname{th} \frac{\pi}{2} \sqrt{\alpha'}}{\left( \frac{\pi}{2} \sqrt{\alpha'} \right)^5}. \end{aligned}$$

La fonction  $\Psi'$  (fig. 11) étant décroissante en partant de l'unité et tendant vers zéro lorsque le paramètre  $\alpha'$  tend vers l'infini. Développement pour de petites valeurs du paramètre:

$$\Psi'(\alpha') = 1 - \frac{17\pi^2}{84} \alpha' + \frac{31\pi^4}{1008} \alpha'^2 - \frac{691\pi^6}{166320} \alpha'^3 + \dots$$

*Raccourcissement vertical  $\lambda'$  en fonction de la flèche horizontale  $f'$ :*

$$\lambda' = \frac{3}{5} \frac{f'^2}{L'} \chi'(\alpha'),$$

avec

$$\chi'(\alpha') = \frac{\Psi'(\alpha')}{\Phi'^2(\alpha')}.$$

La fonction  $\chi'$  (fig. 12 et 7) étant décroissante. Elle vaut l'unité pour la valeur zéro du paramètre et tend vers 0,83333 lorsque le paramètre  $\alpha'$  tend vers l'infini. Développement en série de puissances:

$$\begin{aligned} \chi'(\alpha') = \\ = 1 - \frac{\pi^2}{420} \alpha' + \frac{\pi^4}{25200} \alpha'^2 - \frac{37\pi^6}{58212000} \alpha'^3 + \dots \end{aligned}$$

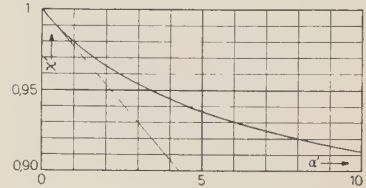


Fig. 12.

### Moment fléchissant aux extrémités.

En fonction de la force horizontale  $F'_0$ :

$$M' = \frac{F'_0 L'}{2} \tau'(\alpha'),$$

avec

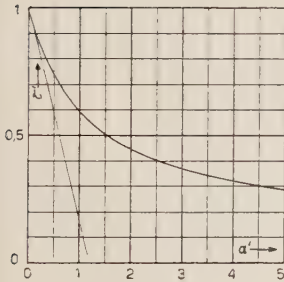


Fig. 13.

$$\tau'(\alpha') = \frac{\text{th } \frac{\pi}{2} \sqrt{\alpha'}}{\frac{\pi}{2} \sqrt{\alpha'}}.$$

La fonction  $\tau'$  (fig. 13) étant décroissante, valant l'unité pour  $\alpha' = 0$  et tendant vers zéro lorsque  $\alpha'$  tend vers l'infini. Développement pour de petites valeurs de  $\alpha'$ :

$$\tau'(\alpha') = 1 - \frac{\pi^2}{12} \alpha' + \frac{\pi^4}{120} \alpha'^2 - \frac{17\pi^6}{20160} \alpha'^3 + \dots$$

En fonction de la flèche horizontale  $f'$ :

$$M' = \frac{6E'I'}{L'^2} \frac{\tau'(\alpha')}{\Phi'(\alpha')}.$$

La fonction  $\tau'/\Phi'$  étant représentée à la fig. 14. Pour de petites valeurs de  $\alpha'$  on a:

$$\frac{\tau'}{\Phi'} = 1 + \frac{\pi^2}{60} \alpha' - \frac{\pi^4}{8400} \alpha'^2 + \frac{\pi^6}{756000} \alpha'^3 - \dots$$



Sur tous les diagrammes tracés on a fait figurer la tangente à l'origine afin de pouvoir estimer grossièrement de combien l'on s'écarte de la loi linéaire obtenue en ne conservant de chaque fonction que les deux premiers termes.

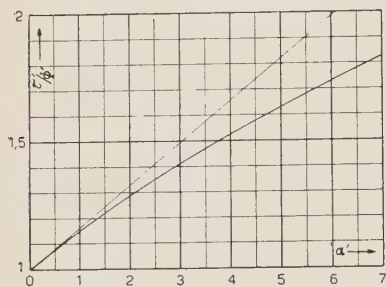


Fig. 14.

En possession des outils précédents permettant de calculer rapidement les éléments de suspension, nous allons pouvoir étudier avec grande facilité les systèmes compensés symétriquement chargés et tirer quelques conclusions générales concernant ces suspensions.

Étudions d'abord les systèmes compensés où les longueurs de lames  $L$  et  $L'$  sont égales. Ce sont les systèmes qui ont été les plus utilisés car ils sont les plus simples à imaginer. Si les lames des deux suspensions sont identiques il ne peut y avoir compensation rigoureuse que dans le cas idéal où les poids sont nuls. C'est un cas limite que nous excluons toujours. Si les lames de suspension ne sont pas identiques, la compensation rigoureuse a lieu dans des conditions que nous allons préciser.

### Systèmes compensés non asservis à lames d'égales longueurs.

Désignons par  $f$  la flèche horizontale de la lame en compression; par  $f'$  la flèche correspondante de la lame en traction; ces deux flèches étant déterminées pour la même valeur de la force horizontale  $F_0$ . Le déplacement  $\xi$  de la platine, égal à  $f+f'$  vaut:

$$\xi = \frac{F_0 L^3}{12} \left[ \frac{\Phi(\alpha)}{EI} + \frac{\Phi'(\alpha')}{E'I'} \right].$$

Quant à la descente  $\eta_c$  de la platine par rapport au bâti elle vaut la différence des raccourcissements verticaux  $\lambda$  et  $\lambda'$  calculés précédemment; d'où:

$$\eta_c = \frac{L^5 F_0^2}{240} \left[ \frac{\Psi(\alpha)}{(EI)^2} - \frac{\Psi'(\alpha')}{(E'I')^2} \right],$$

L'élimination de  $F_0$  entre ces deux relations donne:

$$\eta_c = \frac{3}{5} \frac{\xi^2}{L} \frac{\frac{\Psi(\alpha)}{(EI)^2} - \frac{\Psi'(\alpha')}{(E'I')^2}}{\left[ \frac{\Phi(\alpha)}{EI} + \frac{\Phi'(\alpha')}{E'I'} \right]^2},$$

en particulier, si  $E'I' = EI$ :

$$\eta_c = \frac{3}{5} \frac{\xi^2}{L} \frac{\Psi(\alpha) - \Psi'(\alpha')}{[\Phi(\alpha) + \Phi'(\alpha')]^2}.$$

La discussion de ces formules est immédiate. Si les lames sont identiques, il ne peut y avoir compensation rigoureuse; la trajectoire parasite dans le plan vertical est en dos d'âne. En effet, ayant toujours pour  $\alpha$  et  $\alpha'$  différents de zéro:

$$\Psi(\alpha) > 1 \quad \text{et} \quad \Psi'(\alpha') < 1$$

on a:

$$\Psi(\alpha) - \Psi'(\alpha') > 0;$$

il y a donc descente de la platine.

Si les lames ne sont pas identiques, la trajectoire dépend de la valeur de:

$$\frac{\Psi(\alpha)}{(EI)^2} - \frac{\Psi'(\alpha')}{(E'I')^2},$$

qui est à la disposition du constructeur.

En particulier si on choisit les moments d'inertie des lames de même matière, de telle manière que:

$$\frac{I'}{I} = \sqrt{\frac{\Psi'(\alpha')}{\Psi(\alpha)}} < 1,$$

il y aura compensation rigoureuse. Cette équation est à résoudre par tâtonnements car  $\Psi'(\alpha')$  dépend de  $I'$ .

Le second membre de la relation précédente étant toujours inférieur à l'unité il y aura lieu d'utiliser une suspension en U moins rigide que la suspension de la platine.

Tout changement du moment d'inertie  $I'$  obtenu par modification d'épaisseur de la lame est fort brutal; le moment d'inertie variant comme le cube de l'épaisseur. C'est pourquoi il sera souvent préférable d'agir plutôt sur la largeur de la lame.

Pour la facilité de la construction, il est préférable de choisir les lames de telle manière que le rapport  $I'/I$  soit légèrement inférieur à:

$$\sqrt{\frac{\Psi'(\alpha')}{\Psi(\alpha)}}.$$

La trajectoire de la platine présente alors sa concavité vers le haut. (La platine monte aux extrémités de course). On parfera le réglage en diminuant

$\Psi'(\alpha')$  par augmentation de  $\alpha' = (P'L^2)/(\pi^2 E'I')$ . Cela se fait avec toute la facilité et la sensibilité voulue en augmentant  $P'$  par addition de poids de réglage au cadre raidisseur. C'est une solution simple, peu coûteuse, laissant toute liberté pour l'équilibrage des lames entre elles. Ce procédé a donc l'avantage de dissocier les différents réglages à effectuer.

### Systèmes compensés asservis à lames d'égales longueurs.

Ici, du fait de l'asservissement, les déplacements horizontaux des lames sont égaux. Le déplacement  $\xi$  de la platine vaut donc

$$\xi = 2f.$$

Quant à la descente  $\eta_a$  de la platine par rapport au bâti, elle vaut la différence des raccourcissements verticaux  $\lambda$  et  $\lambda'$  exprimés en fonction du déplacement horizontal, d'où :

$$\eta_a = \frac{3}{20} \cdot \frac{\xi^2}{L} [\chi(\alpha) - \chi'(\alpha')],$$

que les moments d'inertie soient les mêmes ou non.

Ayant :

$$\chi(\alpha) > 1 \quad \text{et} \quad \chi'(\alpha') < 1$$

on en déduit que la compensation n'est jamais parfaite et que la trajectoire parasite est toujours en dos d'âne.

Il est intéressant de comparer la trajectoire parasite du dispositif asservi à celle du dispositif non asservi lorsque les moments d'inertie des lames sont égaux.

Ayant :

$$\eta_c = \frac{3}{5} \cdot \frac{\xi^2}{L} \frac{\Psi(\alpha) - \Psi'(\alpha')}{[\Phi(\alpha) + \Phi'(\alpha')]^2},$$

et :

$$\eta_a = \frac{3}{20} \cdot \frac{\xi^2}{L} [\chi(\alpha) - \chi'(\alpha')],$$

on en déduit que :

$$\frac{\eta_c}{\eta_a} = \frac{4[\Psi(\alpha) - \Psi'(\alpha')]}{[\Phi(\alpha) + \Phi'(\alpha')]^2 [\chi(\alpha) - \chi'(\alpha')]}.$$

Ce rapport est évidemment fonction des deux variables réduites  $\alpha$  et  $\alpha'$ . Remplaçant les diverses fonctions qui y interviennent par leurs développements

en série calculés précédemment, on trouve facilement que pour  $\alpha$  et  $\alpha'$  tendant vers zéro, le rapport  $\eta_c/\eta_a$  tend vers la valeur 85. La réduction de l'importance de la trajectoire parasite, due à l'asservissement est donc très importante.

On trouvera à la fig. 15 un diagramme donnant le rapport  $\eta_c/\eta_a$  en fonction du paramètre  $\alpha$  pour différentes valeurs du rapport des paramètres  $\alpha'/\alpha$ . On y constate que l'influence de l'asservissement reste toujours fort importante même pour des valeurs non négligeables des paramètres  $\alpha$  et  $\alpha'$ . C'est dans ce diagramme que se trouve la justification des systèmes de suspension à asservissement, qui marquent un net progrès sur les suspensions non asservies; progrès obtenu malheureusement au prix d'une complication importante et relativement coûteuse.

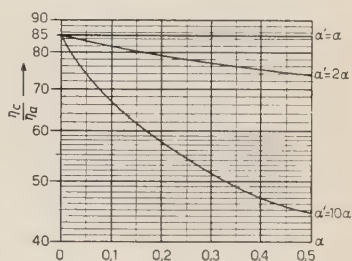


Fig. 15.

Il est vrai que sous cette forme, les dispositifs asservis ne peuvent être complètement compensés par un choix judicieux des moments d'inertie, alors que les dispositifs non asservis jouissent de cette propriété. Nous ne considérons pas ceci comme un vice de la suspension à asservissement, car, comme nous le montrerons plus loin, une compensation rigoureuse peut être obtenue en utilisant des lames de longueurs différentes.

D'ailleurs, si l'on désire utiliser des lames de même hauteur, une compensation rigoureuse peut être obtenue par un choix correct de l'asservissement. En effet,  $f$  désignant la flèche horizontale de la lame comprimée et  $f'$  celle de la lame tendue, réalisons un asservissement tel que  $f' = kf$ ; c'est-à-dire que l'avancement de la platine est à celui du cadre raidisseur comme  $1+k$  est à  $k$ .

On trouve facilement que :

$$\eta_a = \frac{3}{5} \frac{\xi^2}{L(1+k)^2} [\chi(\alpha) - k^2 \chi'(\alpha')].$$

Choisissant  $k$  tel que :

$$k = \sqrt{\frac{\chi(\alpha)}{\chi'(\alpha')}} > 1,$$

on réalise une compensation rigoureuse. On trouve facilement que  $k$  doit être supérieur à l'unité. On choisira une valeur de  $k$  légèrement supérieure à sa valeur optimum et l'on finira la compensation en ajoutant des masses additionnelles au cadre raidisseur comme précédemment.

Un cas particulier intéressant est celui d'une suspension asservie à lames de même hauteur très peu chargées. Les paramètres  $\alpha$  et  $\alpha'$  étant très petits,



on peut dans ce cas se limiter aux deux premiers termes des développements en série des fonctions  $\chi$  et  $\chi'$ :

$$\chi = 1 + \frac{\pi^2}{420} \alpha, \quad \chi' = 1 - \frac{\pi^2}{420} \alpha',$$

d'où:

$$\eta_a = \frac{3}{20} \cdot \frac{\xi^2}{L} \cdot \frac{\pi^2}{420} (\alpha + \alpha');$$

or,

$$\alpha = \frac{PL^2}{\pi^2 EI}, \quad \alpha' = \frac{P'L^2}{\pi^2 EI'},$$

d'où:

$$\eta_a = \frac{\xi^2 L}{2800 E} \left( \frac{P}{I} + \frac{P'}{I'} \right),$$

qui est la formule que nous avons trouvée dans un travail précédent <sup>(2)</sup> en utilisant les formules de Navier-Bresse.

Le présent travail permet d'obtenir des valeurs précises concernant le domaine d'application des formules approchées trouvées dans ce précédent travail. Remarquons qu'ainsi s'éclaire la bizarrerie apparente de la formule précédente où le mouvement parasite semble être proportionnel à la longueur des lames. Il ne faudrait pas en conclure un peu rapidement qu'il n'y a qu'à réduire les hauteurs de lames pour diminuer l'importance du mouvement parasite!

### Suspensions compensées non asservies à lames de longueurs inégales.

Ces suspensions s'étudient aussi sans difficulté. Désignons par  $f$  la flèche horizontale de la lame de platine et par  $f'$  celle de la lame de cadre raidisseur. Le déplacement horizontal  $\xi$  de la platine vaut  $f + f'$  ou encore:

$$\xi = \frac{F_0}{12EI} [L^3 \Phi(\alpha) + L'^3 \Phi'(\alpha')],$$

en supposant  $EI = E'I'$ .

La descente  $\eta_c$  de la platine est égale à la différence des raccourcissements verticaux  $\lambda$  et  $\lambda'$  calculés précédemment:

$$\eta_c = \frac{1}{240} \left( \frac{F_0}{EI} \right)^2 [L^5 \Psi(\alpha) - L'^5 \Psi'(\alpha')].$$

Éliminant  $F_0$  entre les deux relations précédentes on obtient :

$$\eta_c = \frac{3}{5} \xi^2 \frac{L^5 \Psi(\alpha) - L'^5 \Psi'(\alpha')}{[L^3 \Phi(\alpha) + L'^3 \Phi'(\alpha')]^2}.$$

La compensation sera rigoureuse si l'on a :

$$\frac{L'}{L} = \sqrt[5]{\frac{\Psi(\alpha)}{\Psi'(\alpha')}}.$$

Ayant toujours

$$\Psi(\alpha) > 1 \quad \text{et} \quad \Psi'(\alpha') < 1$$

on aura donc  $L' > L$ . La suspension de la platine sera plus courte que celle du cadre raidisseur. En pratique on constate que la différence  $L' - L$  est petite. Ceci explique la forte dispersion rencontrée lors de la comparaison des trajectoires parasites de suspensions du même type. Cette dispersion est imputable à de faibles différences dans les longueurs de lames.

Un ajustement précis de la suspension sera, ici comme précédemment, obtenu en choisissant  $L'$  légèrement plus grand que la valeur optimum et en surchargeant le cadre raidisseur.

Si les moments d'inertie des lames ne sont pas identiques, la relation précédente entre  $L$  et  $L'$  est remplacée par :

$$\frac{L'^5}{L^5} \cdot \frac{I^2}{I'^2} = \frac{\Psi(\alpha)}{\Psi'(\alpha')},$$

que l'on discute de la même manière.

### Suspensions compensées asservies à lames de longueurs inégales.

Limitons-nous au cas où l'asservissement impose des déplacements de platine et de cadre raidisseur dans le rapport de deux à un. Les déformations horizontales des lames sont donc égales et le déplacement horizontal  $\xi$  de la platine a pour valeur :

$$\xi = 2f.$$

La descente  $\eta_a$  de la platine par rapport au bâti est égale à la différence des raccourcissements verticaux  $\lambda$  et  $\lambda'$  exprimés en fonction de la déformation

horizontale des lames, soit :

$$\eta_a = \frac{3}{20} \xi^2 \left[ \frac{\chi(x)}{L} - \frac{\chi'(x')}{L'} \right].$$

La trajectoire sera rigoureusement rectiligne si :

$$\frac{L'}{L} = \frac{\chi'(x')}{\chi(x)} < 1.$$

Il faut donc ici que la longueur de la suspension du cadre raidisseur soit plus petite que celle de la platine. Ici encore, comme précédemment, on choisira  $L'$  très légèrement inférieur à sa valeur optimum et l'on parfera la compensation en chargeant le cadre raidisseur de masses de réglage. On ne perdra pas de vue que la longueur  $L'$  ne peut être choisie beaucoup trop petite car, lorsqu'on augmente indéfiniment la charge du cadre raidisseur  $\chi'(x')$  ne deviendra pas inférieure à 0,8333.

### Conclusions.

Notre étude s'est bornée aux suspensions à lames élastiques sur toute leur hauteur. Il serait intéressant de voir si les conclusions que nous avons trouvées sont encore valables lorsqu'on utilise des lames raidies sur une partie de la hauteur. Cette étude serait malheureusement assez lourde comme on peut le supposer après avoir vu le cas simple que nous avons étudié dans un autre travail déjà cité.

De notre travail il résulte qu'une compensation rigoureuse peut être obtenue facilement tant pour les suspensions asservies que pour celles qui ne le sont pas. Il va sans dire que ces deux types de suspension ne sont pas équivalents. Si l'on envisage la naissance d'une trajectoire parasite due à une surcharge de la platine, la suspension asservie est à préférer de beaucoup.

Sans doute les compensations « rigoureuses » que nous avons trouvées l'ont été en introduisant des hypothèses très générales concernant la déformation des lames, mais ces hypothèses ne sont pas les plus générales que l'on puisse faire. Aussi faut-il s'attendre à une trajectoire parasite résiduelle faible mais qu'il est peut être possible de mettre en évidence.

Quoi qu'il en soit, nous pensons avoir montré que la solution, trop souvent utilisée dans le passé, consistant à rendre les lames le plus rigide possible pour pouvoir négliger les poids, est à rejeter définitivement comme ne présentant absolument aucun avantage. Cette voie de paresse est même dangereuse, car elle conduit à augmenter anormalement les forces à mettre en jeu par la vis de mesure. Il est vrai que ceux qui l'utilisèrent diminuaient

par ce procédé l'importance du mouvement parasite. Cette diminution de l'erreur du dispositif ne justifie pas l'errement de principe de la solution.

L'auteur tient à remercier le professeur G. P. S. OCCHIALINI de l'intérêt amical qu'il n'a cessé de porter à ce travail.

#### RIASSUNTO (\*)

Il presente lavoro sviluppa un metodo generale di calcolo delle sospensioni elastiche semplici e a compensazione, in cui le guide sono flessibili per l'intera altezza. Si comincia col caso di una guida elementare studiata per mezzo dello sviluppo in serie di Fourier della sua deformata. Il metodo permette, col ricorso ad ipotesi assai generali di determinare l'accorciamento verticale delle guide. Si studiano e si confrontano fra di loro le sospensioni compensate, sia asservite che libere. In particolare, si mettono in evidenza i vantaggi della sospensione asservita. Si tratta il caso in cui le sospensioni elementari sono di altezze differenti. Si propongono diversi nuovi tipi di sospensione che realizzano una traiettoria rettilinea anche se utilizzano una platina pesante. Le modificazioni sono semplici e di facile calcolo. Il loro aggiustaggio si effettua indipendentemente dalle altre regolazioni della sospensione.

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(\*) *Traduzione a cura della Redazione;*



## A Meson Effect in the Dipole Selection-Rule in Selfconjugate Nuclei.

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**Summary.** — We point out an effect by which a well known selection rule for emission of dipole radiation in a transition between two levels having the same charge-parity (or isotopic spin) in selfconjugate nuclei may be weakened. The effect depends on the fact that the total momentum of the nucleons plus the momentum associated with the fluctuations of the mesonic field in a nucleus is a constant of the motion but not the total momentum of the nucleons only; while this last circumstance is assumed in the usual derivation of the above selection rule. The factor of weakening has been calculated and turns out to be of the order (in amplitude)  $(g^2/4\pi)(\mu/Mc^2) \cong U/Mc^2$  where  $U$  is some average potential energy in the nucleus,  $M$  the mass of a nucleon and  $\mu$  the inverse Compton wave length of a meson. This is the same order of magnitude as the one due to Coulomb impurity. If it will be possible with an increasing experimental accuracy to separate at least for some transition the mesonic effect from the Coulomb one, this would furnish a direct experimental proof for the existence of virtual mesons in a nucleus or, in other words for a relativistic (meson) theory of nuclear forces.

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### Introduction.

It is well known that the assumption of equality of the forces (nn) (np) (pp) (charge independence), apart from Coulomb effects leads to the introduction for a system of nucleons of a quantum number  $T$ , the isotopic spin of the system, which is an almost good quantum number; this means that if the Coulomb effects and the neutron-proton mass-difference could be disregarded, under the hypothesis of charge independence, to each eigenstate of the system could be given a definite value of  $T$ .

From the fact that  $T$  has the mathematical properties of a spin variable, it follows, as has been shown by RADICATI <sup>(1)</sup>, that for electromagnetic transitions in nuclei the selection rule  $\Delta T = \pm 1, 0$  must hold.

TRAINOR <sup>(2)</sup> and RADICATI <sup>(1)</sup> have moreover shown that for electric dipole ( $E1$ ) transitions in selfconjugate nuclei ( $N = Z$ ) the more restrictive selection rule  $\Delta T = \pm 1$ , must hold, that is dipole radiation cannot be emitted in a transition between two levels having the same value of  $T$ . KROLL and FOLDY <sup>(3)</sup> have shown that this result is true also under the less restrictive assumption of charge symmetry, in which case the isotopic spin loses its meaning but, for selfconjugate nuclei, the charge parity still is a good quantum number and the previous result has to be expressed by saying that dipole transitions between two levels having the same charge parity are forbidden.

This statement (or the previous one  $\Delta T = \pm 1$ ) is valid however under some restrictions. First of all it has been shown by GELL-MANN and TELEGDI <sup>(4)</sup> that in the derivation of the dipole selection rule in selfconjugate nuclei terms of order  $(fR)^2$  in the amplitude have been neglected, so the selection rule itself is valid apart from terms of this order. Secondly—this is generally a much larger effect—there is the Coulomb interaction (and the much smaller neutron-proton-mass difference) which destroys exact charge independence or charge-symmetry <sup>(5)</sup>.

We want now to point out that there is a third effect which reduces the validity of the dipole rule; the effect is due to the fact that if the forces among nucleons are transmitted by virtual mesons the total momentum of the nucleons is no more a constant of the motion, which fact is assumed in the derivation of the dipole rule. Only the momentum of the nucleons plus meson field is a constant of the motion; this fact introduces some corrections to the dipole rule whose exact value is difficult to estimate, but whose order of magnitude should be the same as the one due to the Coulomb corrections.

The effect we mention here has to be distinguished from the one arising from the meson currents inside the nucleus; also this last effect is present if the forces are transmitted by charged mesons but has no influence whatsoever on the dipole rule because the transitions to which it contributes must necessarily have a  $\Delta T = \pm 1$ ; in other words the effect we consider is present also for hypothetical nucleons interacting only through a neutral meson field.

<sup>(1)</sup> L. A. RADICATI: *Phys. Rev.*, **87**, 521 (1952). Substantially the same results have been obtained by A. GAMBA, R. MATVANO and L. A. RADICATI (*Phys. Rev.*, **87**, 440 (1952)) making use of the symmetric group.

<sup>(2)</sup> L. E. H. TRAINOR: *Phys. Rev.*, **85**, 962 (1952).

<sup>(3)</sup> N. M. KROLL and L. L. FOLDY: *Phys. Rev.*, **88**, 1177 (1952).

<sup>(4)</sup> M. GELL MANN and V. L. TELEGDI: *Phys. Rev.*, **91**, 169 (1953).

<sup>(5)</sup> L. A. RADICATI: *Proc. Phys. Soc.*, A **66**, 139 (1953).

We want at this point to stress that, if it will be possible to separate, at least for a particular transition, the effect we have mentioned from the Coulomb one, this could furnish a direct experimental proof of the presence of virtual mesons inside a nucleus or, otherwise stated, of a relativistic (meson) theory of nuclear forces.

In the following paragraphs we shall show in more detail how this mesonic effect comes out, and we shall attempt a calculation of it.

More precisely in § 1 a derivation of the dipole rule is given in the absence of mesons, which exhibits its dependence on just charge symmetry, without introducing at all the concept of isotopic spin. The modification due to the presence of a meson field coupled to the nucleons will be considered in § 2; from there on we shall confine our attention to a scalar meson field, for reasons of simplicity; it will be shown in § 3 that although the effect vanishes in such case for a deuteron, on account of particular symmetries it does not necessarily vanish for a nucleus with a higher number of nucleons. The simplest selfconjugate nucleus after the deuteron is the  $\alpha$ -particle and in § 4 an evaluation of the effect in this case is given for a neutral scalar meson field. The calculation is repeated in § 5 for the symmetric meson theory; use will then be made of the isotopic spin formalism. A discussion of the present experimental situation is given in § 6. An Appendix contains the main steps of the calculations.

## 1. - The dipole rule.

The interaction term which is responsible for the dipole transitions, is apart from the corrections considered in (4):

$$(1) \quad \mathcal{E} = - \frac{e}{Mc} \sum_i^Z \mathbf{p}_i \cdot \mathbf{A},$$

where  $M$  is the mass of a nucleon,  $\mathbf{A}$  the vector potential of the electromagnetic field, independent in the dipole approximation from  $i$ ,  $\mathbf{p}_i$  the momentum of the  $i$ -th proton,  $Z$  the number of the protons. This term induces dipole transitions between the states of the system characterized by an Hamiltonian:

$$(2) \quad H = H_N + H_{\text{e.m.}},$$

where

$$H_N = T_N + V = \sum_i^A \frac{p_i^2}{2M} + V,$$

$V$  being a translation invariant, charge symmetrical potential energy, and  $H_{\text{e.m.}}$  is the Hamiltonian of the free electromagnetic field. The states in question are classified according to (the number of photons and) the eigenfunctions of  $H_N$ ; two of these will be denoted in the following by  $\varphi_1, \varphi_2$ ; such functions are separately antisymmetrical with respect to  $\eta_1, \dots, \eta_Z$  and  $\eta_{Z+1}, \dots, \eta_A$  where the first  $Z$  coordinates are the space and spin coordinates of the protons and the last  $Z$  the space and spin coordinates of the neutrons.

The Hamiltonian, under the hypothesis of charge symmetry being symmetrical with respect to the simultaneous exchange of  $\eta_1$  with  $\eta_{Z+1}$ ,  $\eta_2$  with  $\eta_{Z+2}$ ,  $\eta_Z$  with  $\eta_A$ , it follows that  $\varphi_1$  and  $\varphi_2$  will be eigenfunctions of the operator  $P$ , the charge parity operator which effects such exchange. It is obvious that  $P^2 = 1$  and consequently an eigenfunction of  $H_N$  has either a plus charge parity (in the following c.p.) or a minus c.p. Moreover  $\varphi_1$  (and  $\varphi_2$ ) being invariant with respect to translations we can put, without loss of generality:

$$(3) \quad \sum_1^A \mathbf{p}_i \varphi_1 = 0.$$

We have therefore

$$\sum_1^Z \mathbf{p}_i \varphi_1 = \frac{1}{2} \left( \sum_1^Z \mathbf{p}_i - \sum_{Z+1}^A \mathbf{p}_i \right) \varphi_1$$

and it follows that the matrix element of  $\mathcal{E}$  is proportional to:

$$(4) \quad \frac{1}{2} \langle \varphi_2 | \sum_1^Z \mathbf{p}_i - \sum_{Z+1}^A \mathbf{p}_i | \varphi_1 \rangle.$$

The operator  $\sum_1^Z \mathbf{p}_i - \sum_{Z+1}^A \mathbf{p}_i$  has a minus c.p. and consequently the transition  $1 \rightarrow 2$  is forbidden unless  $\varphi_1$  and  $\varphi_2$  have opposite c.p..

## 2. - Mesonic corrections.

As stated in the introduction we shall now consider the protons and the neutrons coupled to a (scalar neutral) meson field. Before attacking the problem we want to clear a point. The hamiltonian of the system should now be:

$$(5) \quad H = T_N + H_1 + H_m + \mathcal{E} + H_{\text{e.m.}},$$

where  $H_m$  is the hamiltonian describing the free mesons, and  $H_1$  is the interaction between mesons and nucleons;  $H_1$  does not commute with  $\sum_1^A \mathbf{p}_i$  and



is consequently responsible for the effect considered here; and, at the same time takes the place of the phenomenological  $V$  of eq. (2) in determining the binding of the nucleus. We are faced however with the fact that, apart from the Tamm-Dancoff <sup>(6)</sup> method which presents for this calculation some disadvantages <sup>(7)</sup>, present field theory does not allow one to do calculations for bound states in any simple way. In order to overcome this difficulty we shall add to the Hamiltonian  $H$  (5) again a charge symmetric term  $V$  which determines the binding. It is clear that the added term has just the role of determining the binding and, being traslation invariant, has no effect in disturbing the dipole rule, which, were it not for the coupling  $H_1$  with the meson field should hold as has been shown in section 2. In other words, with our procedure we shall solve the following problem; protons and neutrons are coupled together through a translation invariant potential, and moreover they are coupled to a meson field; if the meson field were not in the dipole rule would be valid; on account of the presence of the field it is weakened and we shall calculate the factor of weakening.

Call now  $\mathcal{H} = H + V$ . A state of  $\bar{H} = \mathcal{H} - \mathcal{E} - H_{e.m.}$  describing a nucleus in some state  $\alpha$ , will be called  $\Psi_\alpha$ , the index  $\alpha$  referring to the energy, angular momentum and all the constants of the system  $\bar{H}$ .

The function  $\Psi_\alpha = \Psi_\alpha(\eta_i, q_k)$  depends on the spin and space coordinates  $\eta_i$  of the nucleons and on the coordinates  $q_k$  of the meson field; it is defined as an eigenstate of  $H$  such that, in the limit  $H_1 \rightarrow 0$  it reduces to the product of an eigenstate  $\varphi_\alpha$  of  $T_N + V$  and of an eigenstate of  $H_m$  with zero mesons present.

It is obvious that we can define now the charge parity operation as that operation, which exchanges all neutrons with all protons and leaves the mesonic coordinates  $q_k$  unaltered.  $\bar{H}$  is invariant under such an operation; moreover  $\bar{H}$  is invariant under the spatial parity operation (in the following s.p.) which exchanges  $\mathbf{x}_i$  in  $-\mathbf{x}_i$  and  $q_k$  in  $q_k$ . The eigenfunctions  $\Psi_\alpha$  may be consequently classified according to the s.p. and c.p. and we shall now show that the matrix element of the electromagnetic interaction  $\mathcal{E}$  between two states  $\Psi_\beta$  and  $\Psi_\alpha$  with the same c.p. is no more zero.

In this case one has in fact, instead of (3):

$$(6) \quad \left( \sum_i^A \mathbf{p}_i + \mathbf{G} \right) \Psi_\alpha = 0,$$

<sup>(6)</sup> S. M. DANCOFF: *Phys. Rev.*, **78**, 382 (1950).

<sup>(7)</sup> We want to be sure, in doing the calculation, of the ortogonality of the two states between which the transition takes place; the Tamm-Dancoff treatment insures this ortogonality if its equations are solved exactly, but does not insure it if the equations are solved approximately as necessary for practical reasons.

where  $\mathbf{G}$  is the momentum of the meson field:

$$\mathbf{G} = -\frac{1}{2} \sum_k \mathbf{k} (p_k q_k + q_k p_k),$$

and it follows that:  $\sum_1^Z \mathbf{p}_i \Psi_\alpha = \frac{1}{2} (\sum_1^Z \mathbf{p}_i - \sum_{Z+1}^A \mathbf{p}_i - \mathbf{G}) \Psi_\alpha$ . The matrix element of  $\mathcal{E}$  is now proportional to:

$$\frac{1}{2} \langle \Psi_\beta | \sum_1^Z \mathbf{p}_i - \sum_{Z+1}^A \mathbf{p}_i - \mathbf{G} | \Psi_\alpha \rangle.$$

If  $\Psi_\beta$  and  $\Psi_\alpha$  have the same c.p. this reduces to:

$$(7) \quad -\frac{1}{2} \langle \Psi_\beta | \mathbf{G} | \Psi_\alpha \rangle \equiv \frac{1}{2} \langle \Psi_\beta | \sum_1^A \mathbf{p}_i | \Psi_\alpha \rangle,$$

as  $\mathbf{G}$  has a positive c.p.. The two forms in (6) are equivalent by virtue of (6).

It is also clear from the form (7) of the matrix element that  $\Psi_\alpha$  and  $\Psi_\beta$  must have opposite spatial parity; and  $\mathbf{G}$  being a vector, angular momenta differing by one or zero, as it must be for dipole transitions.

The result contained in (7) may be intuitively expressed by saying that the « Zitterbewegung » of the center of mass of the nucleons breaks down the dipole rule. Note however that the expectation value of the momentum of the center of mass of the nucleons in a stationary state vanishes, for spatial parity reasons. It is just the matrix element between two different states which does not in general necessarily vanish.

### 3. - The case of the deuteron.

The most simple self conjugate nucleus is the deuteron; in order to get the order of magnitude for the effect we should be tempted to use the deuteron, but the matrix element (7) vanishes exactly in such case if we use a scalar interaction. The reason is the following: suppose (there is no loss of generality in this) that  $\Psi_\alpha$  has s.p. +, c.p. +; moreover, the total spin being a constant of the motion (in the scalar case),  $\Psi_\alpha$  must have some definite spin, either 1 or 0. The state described by  $\Psi_\beta$  must have the same c.p., opposite s.p. and the same spin as  $\Psi_\alpha$ ; in fact  $\mathbf{G}$  in (7) does not contain the spin. Now the operation of changing  $\mathbf{x}_1$  into  $\mathbf{x}_2$  and viceversa, and the operation of changing  $\mathbf{x}_1$  into  $-\mathbf{x}_1$ ,  $\mathbf{x}_2$  into  $-\mathbf{x}_2$  are both equivalent to change  $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$  into  $-\mathbf{x}$ .

Let's now write:

$$\Psi_{\alpha} = \Phi_{\alpha}(\mathbf{x}, q_k) \begin{cases} s_{12} \\ t_{12} \end{cases} \quad \Psi_{\beta} = \Phi_{\beta}(\mathbf{x}, q_k) \begin{cases} s_{12} \\ t_{12} \end{cases},$$

where  $s_{12}$  and  $t_{12}$  denote a singlet or triplet state and either both singlet or both triplet states have to be taken; from the c.p. and s.p. properties of  $\Psi_{\alpha}$  and  $\Psi_{\beta}$  it follows:

$$(8) \quad \begin{cases} \Phi_{\alpha}(\mathbf{x}, q_k) = \mp \Phi_{\alpha}(-\mathbf{x}, q_k), \\ \Phi_{\alpha}(\mathbf{x}, q_k) = \Phi_{\alpha}(-\mathbf{x}, q_{-k}), \end{cases}$$

and

$$(9) \quad \begin{cases} \Phi_{\beta}(\mathbf{x}, q_k) = \mp \Phi_{\beta}(-\mathbf{x}, q_k), \\ \Phi_{\beta}(\mathbf{x}, q_k) = -\Phi_{\beta}(-\mathbf{x}, q_{-k}), \end{cases}$$

the  $-$  or  $+$  sign in (8), (9) applying respectively in the singlet and triplet case. Equations (8) and (9) have to be true also in the limit  $H_1 \rightarrow 0$  and in such case the  $\Phi$  functions must reduce to the product of a state with zero mesons, which is invariant under exchange of  $q_k$  into  $q_{-k}$  and a spatial function of  $\mathbf{x}$ , say  $q(\mathbf{x})$ . According to (8) and (9)  $q_{\alpha}(\mathbf{x})$ ,  $q_{\beta}(\mathbf{x})$  must then satisfy:

$$(10)_1 \quad \begin{cases} q_{\alpha}(\mathbf{x}) = \mp q_{\alpha}(-\mathbf{x}), \\ q_{\alpha}(\mathbf{x}) = + q_{\alpha}(-\mathbf{x}), \end{cases}$$

$$(10)_2 \quad \begin{cases} q_{\beta}(\mathbf{x}) = \mp q_{\beta}(-\mathbf{x}), \\ q_{\beta}(\mathbf{x}) = - q_{\beta}(-\mathbf{x}). \end{cases}$$

If we are dealing with triplet states (10)<sub>2</sub> leads to  $q_{\beta} = 0$ , if we are dealing with singlet states (10)<sub>1</sub> leads to  $q_{\alpha} = 0$ . In both cases it is not possible to construct two wave functions  $\Psi_{\alpha}$  and  $\Psi_{\beta}$  which have a non zero matrix element (7).

The vanishing of the matrix element (7) in this case, depends, as it is apparent from the proof, substantially from the fact that the exchange of  $\mathbf{x}_1$  into  $-\mathbf{x}_1$  and  $\mathbf{x}_2$  into  $-\mathbf{x}_2$  in the expression  $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$  is equivalent to the exchange of  $\mathbf{x}_1$  with  $\mathbf{x}_2$ . There is no such simple relationship in systems of more than two bodies; moreover the proof makes use of the fact that we had to do with a scalar theory; the result would not necessarily be true in a pseudo-

scalar theory (pv. coupling). However the case of a pseudoscalar theory is more complicated and we shall, in the next number evaluate the order of magnitude of the matrix element (7) in the case of a four body system with the scalar theory.

#### 4. - A four body system.

In dealing with such a system we assume for  $V$  in  $H$  the following expression:

$$(11) \quad V = \frac{1}{2} \sigma \sum_{\substack{i, k \\ (i > k)}} (\mathbf{x}_i - \mathbf{x}_k)^2,$$

where  $\sigma$  is a parameter which is related to the extension of the system. Expression (11) has an obvious number of advantages, apart from the disadvantage of giving rise to a non ionizable  $\alpha$ -particle; this has no consequence for our problem; and we shall in the following use simply the denomination  $\alpha$ -particle for our system.

We shall now do a calculation of (7) considering  $\mathcal{E} + H_1$  as a perturbation. This means that the functions  $\Psi_\alpha$  which appear in (7) will be calculated by means of perturbation theory starting from a zero-order approximation as defined generally in no. 3.  $H_1$  may be written

$$(12) \quad H_1 = g e \sum_i^A \varphi(\mathbf{x}_i),$$

with an obvious meaning of the symbols.

It is easily seen that the lowest order non vanishing term is of order  $eg^2$ . One can also realize that, to this order, there is no renormalization to be done. In fact the lowest order mass renormalization counter-term, let's call it  $R$ , is of order  $g^2$  and the  $eg^2$  order perturbation theory formula reads in this case:

$$(13) \quad \sum_I \frac{\langle 2 | \mathcal{E} | I \rangle \langle I | R | 1 \rangle}{E_2 - E_I} + \sum_I \frac{\langle 2 | R | I \rangle \langle I | \mathcal{E} | 1 \rangle}{E_2 - E_I},$$

where  $\langle 2 |$  and  $| 1 \rangle$  are the final and initial states. Expression (13) vanishes on account of the fact that  $\mathcal{E} | 1 \rangle = \langle 2 | \mathcal{E} = 0$  due to the circumstance that the momentum of the center of mass of the nucleons may be put zero in the initial and final states.

We are now in a position to write the matrix element of the transition between two states of our  $\alpha$ -particle. Call  $\varphi_1$  and  $\varphi_2$  the states in question having the same c.p.. The matrix element  $M$  for the process, which one can



derive by simple calculations reads <sup>(8)</sup>:

$$(14) \quad M = g^2 c^2 \sqrt{\frac{\hbar^2 c^2}{2f}} \frac{e}{2Mc} \frac{1}{V} \sum_{\mathbf{k}} \sum_{\alpha} \frac{\hbar^2}{2\omega_{\mathbf{k}}} (\mathbf{e}_f \cdot \mathbf{k}) \cdot \frac{\langle \underline{\varphi}_2 | \sum_i^4 \exp i\mathbf{k} \cdot \underline{\xi}_i | \underline{\varphi}_{\alpha} \rangle \langle \underline{\varphi}_{\alpha} | \sum_i^4 \exp -i\mathbf{k} \cdot \underline{\xi}_i | \underline{\varphi}_1 \rangle}{(E_1 - E_{\alpha} - \hbar\omega_{\mathbf{k}})(E_2 - E_{\alpha} - \hbar\omega_{\mathbf{k}})}$$

In (14)  $V$  is the normalization volume,  $\mathbf{e}_f$  the polarization vector of the photon,  $f$  the photon energy,  $E_1, E_2$  the initial and final energies of the nuclear system ( $E_1 - E_2 = f$ ),  $\underline{\xi}_i = \mathbf{x}_i - \mathbf{X}$ ,  $\mathbf{X} = (\sum_i^4 \mathbf{x}_i)/4$  the coordinates of the nucleons relative to the center of mass. The functions  $\varphi$  have been underlined in order to remember their dependence on the relative coordinates only and the sum is extended over all the meson moment  $\mathbf{k}$  and the nuclear states  $\alpha$ .

It is apparent from (14) that  $M$  vanishes unless the s.p. of  $\varphi_2$  is opposite to that of  $\varphi_1$ . This is best exhibited by writing  $M$  as:

$$(15) \quad M = g^2 c^2 \frac{e}{2Mc} \sqrt{\frac{\hbar^2 c^2}{2f}} \frac{1}{V} \sum_{\mathbf{k}} \sum_{\alpha} \frac{\hbar^2}{2\omega_{\mathbf{k}}} (\mathbf{e}_f \cdot \mathbf{k}) \left\{ \frac{\langle \underline{\varphi}_2 | \sum_i^4 \sin \mathbf{k} \cdot \underline{\xi}_i | \underline{\varphi}_{\alpha} \rangle}{(E_1 - E_{\alpha} - \hbar\omega_{\mathbf{k}})} \cdot \right. \\ \left. \frac{\langle \underline{\varphi}_{\alpha} | \sum_i^4 \cos \mathbf{k} \cdot \underline{\xi}_i | \underline{\varphi}_1 \rangle}{(E_2 - E_{\alpha} - \hbar\omega_{\mathbf{k}})} - \frac{\langle \underline{\varphi}_2 | \sum_i^4 \cos \mathbf{k} \cdot \underline{\xi}_i | \underline{\varphi}_{\alpha} \rangle \langle \underline{\varphi}_{\alpha} | \sum_i^4 \sin \mathbf{k} \cdot \underline{\xi}_i | \underline{\varphi}_1 \rangle}{(E_1 - E_{\alpha} - \hbar\omega_{\mathbf{k}})(E_2 - E_{\alpha} - \hbar\omega_{\mathbf{k}})} \right\},$$

which form exhibits the function in curly brackets as an odd function of  $\mathbf{k}$ .

An evaluation of the order of magnitude of  $M$  can be done as follows: the integrals of the type  $\langle \varphi | \sum_i^4 \sin \mathbf{k} \cdot \underline{\xi}_i | \varphi_{\alpha} \rangle$ ,  $\langle \varphi | \sum_i^4 \cos \mathbf{k} \cdot \underline{\xi}_i | \varphi_{\alpha} \rangle$  which appear in (15) become rapidly zero, whatever the value of  $\alpha$  is, when  $1/|\mathbf{k}|$  is much shorter than the nuclear dimensions  $R$ ; we may thus be justified in cutting the sum over  $\mathbf{k}$  at some maximum  $|\mathbf{k}|$  such that  $|\mathbf{k}|R \cong 1$ ; at the same time developing  $\sum_i^4 \sin \mathbf{k} \cdot \underline{\xi}_i$  inside the integrals in series of  $\mathbf{k} \cdot \underline{\xi}_i$  and stopping at the first non vanishing order.

If we apply this procedure to this neutral case the first non vanishing order is obtained by developing  $\sum_i^4 \sin \mathbf{x} \cdot \underline{\xi}_i$  to the third order; the first order term,

<sup>(8)</sup> In writing the matrix element both now and in the case of the charged theory we shall always disregard a phase factor such as a sign or an imaginary unit.

$\sum_i^4 \mathbf{k} \cdot \boldsymbol{\xi}_i$  vanishes on account of the definition of the  $\boldsymbol{\xi}_i$ ,  $\sum_i^4 \cos \mathbf{k} \cdot \boldsymbol{\xi}_i$  has to be developed to the first order giving 4.

The expression (15) for  $M$  simplifies then to:

$$(16) \quad M = g^2 c^2 \sqrt{\frac{\hbar^2 c^2}{2f}} \frac{e}{2Mc} \frac{\Delta E}{V} \sum_k \frac{\hbar^2}{2\omega_k} \frac{8(\mathbf{e}_f \cdot \mathbf{k})}{(\hbar\omega_k)^3} \frac{1}{3!} \langle \varrho_2 | \sum_i^4 (\mathbf{k} \cdot \boldsymbol{\xi}_i)^3 | \varrho_1 \rangle.$$

The sum over  $\alpha$  has disappeared on account of  $\sum_i^4 \cos \mathbf{k} \cdot \boldsymbol{\xi}_i \rightarrow 4$ ;  $\Delta E$  is the difference  $E_1 - E_2 = f$ .

Our next problem concerns the expressions for the initial and final wave functions. These must satisfy the following conditions: be eigenfunctions of  $H_N$ , have the same c.p., opposite s.p., angular momentum differing by  $\pm 1$  or 0 and the same spin.

Call 1, 2 protons, 3, 4 neutrons; introduce the following normal coordinates:

$$(17) \quad \left\{ \begin{array}{l} \mathbf{q}_1 = \frac{1}{2} [(\mathbf{x}_1 + \mathbf{x}_2) - (\mathbf{x}_3 + \mathbf{x}_4)], \\ \mathbf{q}_2 = \frac{1}{\sqrt{2}} (\mathbf{x}_1 - \mathbf{x}_2), \\ \mathbf{q}_3 = \frac{1}{\sqrt{2}} (\mathbf{x}_3 - \mathbf{x}_4), \\ \mathbf{X}' = \frac{1}{2} (\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4) \equiv 2\mathbf{X}. \end{array} \right.$$

The interaction  $V$  takes the form  $2\sigma(q_1^2 + q_2^2 + q_3^2)$  and  $H_N$  is separable in these coordinates. Two normalized eigenfunctions satisfying our conditions are:

$$(18)_1 \quad \varrho_2 = N_1 N_0^3 \exp \left[ -\frac{1}{2a^2} (q_1^2 + q_2^2 + q_3^2) \right] \cdot \frac{2}{a} q_{1z} \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1)}{\sqrt{2}} \frac{(\alpha_3 \beta_4 - \alpha_4 \beta_3)}{\sqrt{2}},$$

$$(18)_2 \quad \varrho_1 = N_2 N_0^3 \exp \left[ -\frac{1}{2a^2} (q_1^2 + q_2^2 + q_3^2) \right] \cdot 4 \left( \frac{q_2^2}{a^2} - \frac{q_3^2}{a^2} \right) \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1)}{\sqrt{2}} \frac{(\alpha_3 \beta_4 - \alpha_4 \beta_3)}{\sqrt{2}}.$$

Wave function  $\varrho_1$  is an  $S$  one,  $\varrho_2$  a  $P$  one; both belong to the three dimensional  $\begin{smallmatrix} \square & \square & \square \end{smallmatrix}$  representation of the permutation group over four objects; both functions have spin zero.  $N_2, N_1, N_0$  are normalization factors for the hermite functions of order 2, 1, 0;  $a$  is related to the  $\sigma$  of equation (11) by:

$$a = \sqrt[4]{\hbar^2 / 2\sigma M}.$$

We can assume the nuclear radius  $\approx a$  or perhaps a little larger; however it is very probable that the cut-off factor over  $|\mathbf{k}|$  be somewhat larger than  $1/a$ ,

as can be seen considering a typical integral of the kind  $\langle \varphi_2 | \sin \mathbf{k} \cdot \boldsymbol{\xi}_1 | \varphi_1 \rangle$ ; as the present evaluations are supposed to give just an order of magnitude we shall assume for the cut-off factor in the sum over  $|\mathbf{k}|$  the value  $\mu = mc/\hbar$  ( $m$  = meson mass); it is possible that a value double than this should be a better one.

We make now use of:

$$(19) \quad \frac{1}{3!} \langle \varphi_2 | \sum_i (\mathbf{k} \cdot \boldsymbol{\xi}_i)^3 | \varphi_1 \rangle = \frac{4}{3!} \langle \varphi_2 | (\mathbf{k} \cdot \boldsymbol{\xi}_1)^3 | \varphi_1 \rangle = \frac{a^3 k^2 k_z}{4},$$

and can now write  $M$  as:

$$(20) \quad \frac{1}{2\pi} \frac{e}{c} \sqrt{\frac{\hbar^2 c^2}{2f}} \frac{\Delta E}{\hbar} a \cdot \frac{g^2}{4\pi\hbar c} \frac{\hbar}{Mc} \frac{1}{a} \cdot \frac{2}{3} (a\mu)^3 \int_0^1 \frac{x^6 dx}{(x^2 + 1)^2}.$$

In this expression the factor:

$$(21) \quad \frac{1}{2\pi} \frac{e}{c} \sqrt{\frac{\hbar^2 c^2}{2f}} \frac{\Delta E}{\hbar} a,$$

has just the correct energy dependence for a typical dipole matrix element if the transition were completely unprohibited. With the factor  $1/2\pi$  which we have on purpose included in it, it corresponds to a width:

$$(22) \quad \Gamma_\gamma \cong 0.014 f^3 \quad (\Gamma_\gamma \text{ in eV, } f \text{ in MeV}),$$

which is typical for nuclear dipole allowed transitions; compare on this point the discussion in § 6.

The remaining factor, leaving aside a numerical factor:

$$(23) \quad \frac{2}{3} (a\mu)^3 \int_0^1 \frac{x^6 dx}{(x^2 + 1)^2} \sim \frac{1}{9},$$

is:

$$(24) \quad Q_m = \frac{g^2}{4\pi\hbar c} \frac{\hbar}{Mc} \frac{1}{a} = \frac{1}{1.4} \frac{g^2}{4\pi\hbar c} \frac{m}{M} \cong 3 \cdot 10^{-2},$$

where  $m$  is the meson mass and we have made use of  $a = 4^{1/2}(\hbar/mc)$ .

Formula (24) is probably more general than suggests the procedure by which it has been derived. Note that it can also be written as:

$$(25) \quad Q_m = \frac{1}{1.4} \frac{g^2 \mu}{4\pi} \frac{1}{Mc^2} \cong \frac{U}{Mc^2}.$$

In the right side of (25) we have indicated by  $U$  some average value of the potential energy in the nucleus and the result can be expressed by saying that the effect here considered is of the order of the ratio between the average potential energy between a pair of nucleons and the rest mass of a nucleon.

There is finally to discuss the numerical factor (23). The numerical factor is in fact sufficiently small but the procedure we have used in deriving formula (20) is such that we cannot attribute much meaning to it. First of all the functions  $(18)_1$  and  $(18)_2$  have been chosen for reasons of simplicity and they probably don't overlap so as to give the largest possible value of the expression (19). Second the value of the integral in (23) is very critical with respect to the cut-off radius. The precise value of this is very doubtful, as it has been already discussed; we remark that if we had cut at a  $|\mathbf{k}| = 2\mu$  we would have obtained for the numerical factor 3.2; however such a cut-off factor  $|\mathbf{k}| = 2\mu$  would be probably too large for the series in  $|\mathbf{k}|$  to converge.

Thirdly we shall see that in the charged theory the numerical factor is substantially larger (more than 3 times) than (23) due to the fact that the first order term in the  $\mathbf{k}$  development is of order  $k$  and not  $k^2$ ; this implies a factor  $x^4$  instead of  $x^6$  in the numerator of the integrand in (23).

We quote here for ease of comparison the order of magnitude of the factors arising from the Coulomb mixture of levels with different charge parities; we have for the case of low energy levels of light nuclei:

$$(26) \quad \varrho_c = \frac{1}{2} \frac{e^2/r_0}{E^+ - E^-} \cong 2.5 \cdot 10^{-2}. \quad (r_0 = \text{classical electron radius})$$

In (26)  $E^+ - E^-$  is the difference between neighbouring levels with opposite charge parities. The figure given in (26) is just a dimensional order of magnitude, in the same way as (24) or (25) can be considered an order of magnitude for the mesonic effect. RADICATI has shown, by calculating with the shell model several cases that the estimate (26) has sometimes to be multiplied by a numerical factor, of the order  $10^{-1}$ .

We may conclude by saying that, as order of magnitude, the mesonic effect seems to be as large as the Coulomb one; it does not seem possible at the present stage of meson and nuclear theory to say more than this.

## 5. — Symmetrical meson theory.

We shall give here a rapid evaluation of the effect for the case of charged and neutral mesons combined in a symmetrical (Kemmer) theory.

The main steps of these calculations can be found in the Appendix. We shall make now use of the complete formalism of isotopic spin.



$H_1$  is in this case:

$$(27) \quad H_1 = cg' \sum_i^A \{ \sqrt{2} (\tau_i^+ \varphi^*(\mathbf{x}_i) + \tau_i^- \varphi(\mathbf{x}_i)) + \tau_i^{(3)} \varphi_3(\mathbf{x}_i) \},$$

with an obvious meaning of the symbols. We have now, besides the interaction term (1) between electromagnetic field and protons, another term which represents an interaction between photons and mesons. Such term gives no contribution to the transitions between levels with the same  $T$  because it behaves, with respect to rotations in isotopic spin space, as the third component of a vector <sup>(9)</sup>.

The wave functions  $(18)_1$  and  $(18)_2$  have to be completed so as to use the formalism of isotopic spin; this can be done by writing them as follows:

$$(28)_1 \quad \tilde{q}'_2 = \frac{1}{4\sqrt{6}} A \tilde{q}_2 (\eta_1 \eta_2 \eta_3 \eta_4) u_1 u_2 v_3 v_4,$$

$$(28)_2 \quad \tilde{q}'_1 = \frac{1}{4\sqrt{6}} A \tilde{q}_1 (\eta_1 \eta_2 \eta_3 \eta_4) u_1 u_2 v_3 v_4,$$

where  $u_i$  are the isotopic spin functions for a proton and  $v_i$  for a neutron;  $A$  means antisymmetrization with respect to 1, 2, 3, 4 and  $1/4\sqrt{6}$  is a normalization factor. Function  $(28)_1$  (and an exactly similar form is valid for function  $(28)_2$ ) can be written in the form:

$$(29) \quad \frac{1}{\sqrt{6}} [(12) - (34)]_z (1234) \tilde{1} \tilde{2} \tilde{3} \tilde{4} [u_1 u_2 v_3 v_4 - u_3 u_4 v_1 v_2] - \\ - \frac{1}{\sqrt{6}} [(13) - (24)]_z (1234) \tilde{1} \tilde{3} \tilde{2} \tilde{4} [u_1 u_3 v_2 v_4 - u_2 u_4 v_1 v_3] - \\ - \frac{1}{\sqrt{6}} [(14) - (32)]_z (1234) \tilde{1} \tilde{4} \tilde{3} \tilde{2} [u_1 u_4 v_3 v_2 - u_3 u_2 v_1 v_4],$$

where we have put:

$$[(12) - (34)]_z (1234) \tilde{1} \tilde{2} \tilde{3} \tilde{4} = N_1 N_0^8 \frac{(\mathbf{x}_1 + \mathbf{x}_2)_z - (\mathbf{x}_3 + \mathbf{x}_4)_z}{a} \exp \left[ - \frac{1}{8a^2} \sum_{i,k} (\mathbf{x}_i - \mathbf{x}_k)^2 \right] \cdot \\ \cdot \frac{(\alpha_1 \beta_2 - \alpha_2 \beta_1)}{\sqrt{2}} \cdot \frac{\alpha_3 \beta_4 - \alpha_4 \beta_3}{\sqrt{2}},$$

<sup>(9)</sup> K. WATSON: *Phys. Rev.*, **85**, 852 (1952).

which is just, in a more transparent notation, the function  $\varphi_2(1234)$ . It is easy to recognize that the expression  $u_1 u_2 v_3 v_4 - u_3 u_4 v_1 v_2$  and the similar ones appearing in (29) are just eigenfunctions of the isotopic spin of a four nucleon problem, belonging to a value of  $T = 1$  (and  $T_z = 0$ ). So we may say that, besides the properties already given, the functions  $(28)_1$  and  $(28)_2$  have both an isotopic spin 1.

The rest of the analysis is straightforward and follows closely the analysis given for the neutral case. Some details are given in the appendix. Use has been made of the identity (when operating on a completely antisymmetrical function):

$$(30) \quad 2(\tau_i^+ \cdot \tau_j + \tau_i^- \cdot \tau_j^+) + \tau_i^{(3)} \cdot \tau_j^{(3)} = \tau_i \cdot \tau_j = \begin{cases} 3 & (i = j) \\ -1 - 2P_{ij}^H & (i \neq j), \end{cases}$$

where  $P_{ij}^H$  is the Heisenberg space-spin exchange operator.

The matrix element for the process has again the same form as the one (20), the numerical coefficient only being different; the expansion in  $k$  begins in this case with a first order non vanishing term and as a consequence the numerical coefficient for the present case is much larger than in the neutral case; instead of (23) we have now

$$\frac{8}{3} (a\mu) \int_0^1 \frac{x^4 dx}{(x^2 + 1)^2} \sim 0.3.$$

## 6. - The experimental situation.

The experimental situation seems presently to be the following: there is definite evidence that  $EI$  transitions between levels with the same  $T$  in  $T_z = 0$  nuclei are much reduced, with respect to the strength they would have if they were allowed; this statement although qualitatively correct, suffers however seriously, from a more quantitative standpoint, from the fact that this last strenght is very difficult to evaluate. Its value is in fact usually based on the Weiskopf formula; such formula can be substantially <sup>(10)</sup> obtained assuming for the matrix element for the dipole transition the expression (21) without the  $1/2\pi$  factor and may be written concisely:

$$(31) \quad \Gamma_\gamma = 0.1 f^3 A^{\frac{2}{3}} \quad (\Gamma_\gamma \text{ in eV, } f \text{ in MeV}).$$

<sup>(10)</sup> Apart from a factor of 5.

WILKINSON<sup>(11)</sup> has shown that, if the theoretical width is calculated by equation (31) then the ratio  $|M|^2$  between the experimental and the theoretical width, multiplied by  $2J+1$  (where  $J$  is the angular momentum of the initial state), does not show any correlation with the energy for several  $\gamma$  transitions, this showing that the  $f^3$  dependence of (31) is correctly given. However  $|M|^2$

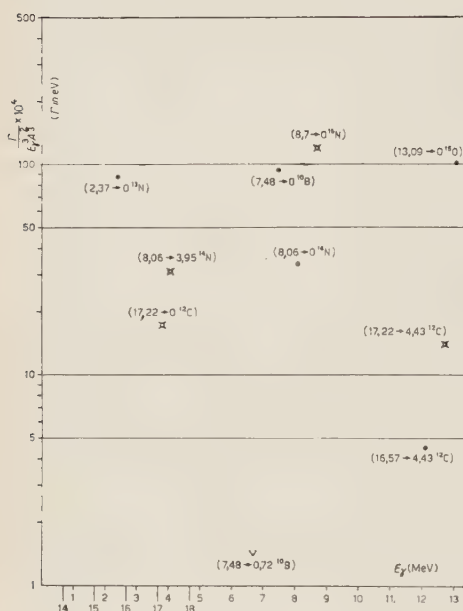


Fig. 1. — Widths for normal E.I. transitions. Abscissae: photon energy  $E_\gamma$  in MeV. Ordinates:  $(\Gamma/E_\gamma^3 A^3) \cdot 10^4$ , where  $\Gamma$  is the width for the transition expressed in eV and  $A$  the atomic number; the ordinate-scale is logarithmic. This figure is simply a graphical reproduction of the data contained in table I of the paper by LANE and RADICATI<sup>(12)</sup>. The  $\bullet$  points correspond to an assignment of the spins parities and  $T$  of the initial and final states which is certain, the  $\times$  correspond to an uncertain assignment of part of the above quantities. The numbers in parenthesis near each point are the energy levels between which the transition takes place.

comes out to be very small this showing that expressions (31) (or what substantially is the same (21) without the  $1/2\pi$  factor) overestimates the transition probabilities. From a table (compare also Fig. 1) given in a paper by LANE and RADICATI<sup>(12)</sup>  $|M|^2$  (1.2A in the notation of LANE and RADICATI for the A region we are dealing with) is almost always smaller than 0.1, and a value of 0.01 is by no means an exception. This means that in the comparison of the radiative widths for our  $T \rightarrow T$  dipole transitions with the radiative width of an unprohibited transition, one cannot assume for the latter the value as calculated by formula (31). Formula (12) is preferable. WILKINSON uses sometimes  $\Gamma_\gamma = 0.02A^{\frac{2}{3}}f^3$ . What is more important is that this conclusion is probably valid not only for the absolute values of the EI intensities, but also for the ratios, say between the intensity of a EI and transitions of another type. In fact the values of  $|M|^2$  for MI transitions, although very

(11) D. H. WILKINSON: *Phil. Mag.*, **44**, 450 (1953).

(12) A. M. LANE and L. A. RADICATI: *Proc. Phys. Soc. London*, **67**, 167 (1954).

scattered, have, the most part, a value around one (in fact they vary from 0.02 to 11.1), compare also Fig. 2.

LANE and RADICATI have tried to give more reliable values of  $M^2$  on the basis of the shell model but their conclusion is that neither the L.S. nor the J.J. coupling gives good agreement with the experience, although some intermediate coupling may well do so (see their Table I).

This being the situation, quantitative determinations of the factors of discouragement are very difficult. Consider for example the case of  $^{16}\text{O}$  given by JONES and WILKINSON <sup>(13)</sup>; consider the two lines

$$\gamma_1 \equiv (E1 \ 7.116(T=0) \rightarrow \text{ground. st.} \\ (T=0))$$

and

$$\gamma_2 \equiv (E2 \ 7.116 \rightarrow 6.137) ;$$

according to the results of JONES and WILKINSON there are surely more than  $(1/8) \cdot 10^3 \gamma_1$  transitions for every  $\gamma_2$  transition. According to formula (26) and to a similar one for E2 transitions this ratio should be of the order  $2.4 \cdot 10^7$ . This means that the rule has discouraged the transition by less than a factor

$(2.4 \cdot 10^7) / ((1/8) \cdot 10^3) \cong 2 \cdot 10^5$ ; according to WILKINSON this implies that the Coulomb contamination of the upper state is higher than  $2 \cdot 10^{-3}$  in amplitude. GELL-MANN and TELEGDI have observed that the relativistic corrections could however explain the observed intensity also without assuming any Coulomb impurity. It is clear that also the effect here proposed could, by itself, explain the transition. Suppose now however to assume that the theoretical ratio between  $\gamma_1$  and  $\gamma_2$  should be  $10^5$  instead than  $2.4 \cdot 10^7$ ; this does not seem impossible, although perhaps improbable. Then WILKINSON's result is that the rule has discouraged the transition by less than a factor  $10^3$ . This is no more explainable on the basis of GELL-MANN and TELEGDI's correction

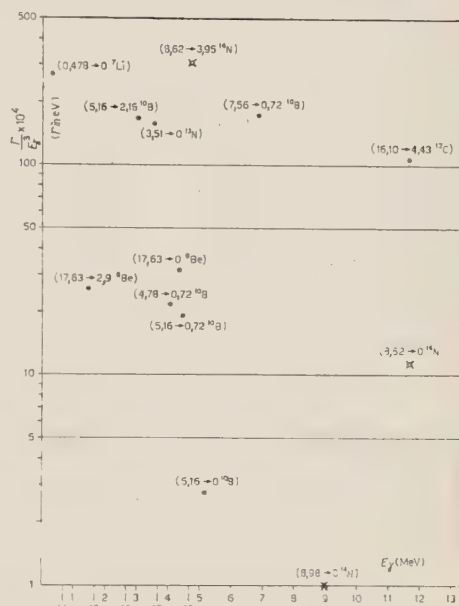


Fig. 2. - Widths for M.I. transitions. Abscissae: photon energy in MeV. Ordinates:  $(\Gamma/E_\gamma^3) \cdot 10^5$ ; the same meaning of the symbols as in Fig. 1. This figure is a graphical reproduction of the data contained in table II of the paper by LANE and RADICATI <sup>(12)</sup>.

<sup>(13)</sup> A. B. CLEGG and D. M. WILKINSON: *Phil Mag.*, **44**, 1269, 1322 (1953).



by a factor  $\sim 100$ ; and the Coulomb impurity it would imply is higher in amplitude than  $1/\sqrt{10^3} \cong 3 \cdot 10^{-2}$ ; would such a value of the Coulomb impurity be possible? According to the calculations by RADICATI<sup>(5)</sup> we are just on the limit.

We have discussed in some detail this case because it is somewhat typical of the kind of discussions one encounters in these problems. WILKINSON has also studied the dipole rule, with reference to other cases<sup>(14)</sup>; in particular with reference to the 8.06 and 2.13 MeV  $T=1$  levels in  $^{14}\text{N}$  concluding that the ratio between the intensity of this transition to the one (also  $E1$ ) 8.06 MeV  $\rightarrow g.s.t$  ( $T=0$ ) is less than (or perhaps equal to) 0.7 which implies a Coulomb impurity of the 8.06 and 2.31 levels of less than (or perhaps equal to)  $2 \cdot 10^{-2}$  in intensity; if the equality is true this would be a very high Coulomb impurity; perhaps a direct method to determine this impurity would be to observe the inelastic scattering of deuterons by  $^{14}\text{N}$ .

We can conclude by saying that although the effectiveness of the dipole rule is established without doubt, especially by the works of WILKINSON, it will be necessary to wait for more extensive experimental data and in particular for precision measurements, to separate clearly the effects by which the rule is weakened. To this purpose further calculations on Coulomb impurities, of the type performed by RADICATI should be very useful.

I want to express my best thanks to Prof. B. FERRETTI for an useful discussion.

## APPENDIX

### Calculation of $M$ and $M'$ .

It is convenient to use the interaction representation; the expression to be evaluated is a third order one and the only non vanishing term is<sup>(8)</sup>:

$$(A1) \quad \mathcal{M}' = \hbar^{-3} \langle \varphi'_1 0_k 1_f | H_1(t) \mathcal{E}(t') H_1(t'') | 0_f 0_k \varphi'_2 \rangle,$$

where integration has to be done over the times  $t, t', t''$ , and the initial and final states represent respectively: no mesons; no photon, and nucleus in state  $\varphi'_1$  (energy  $E_1$ ); no mesons, one photon, and nucleus in state  $\varphi'_2$  (energy  $E_2$ ). Due to the commutativity of  $H_N$  with the  $\tau$  operators appearing in  $H_1$  (27), these can be carried through the integrand to the extreme right. We now make use of  $a_k(t) = a_k \exp[-(it/\hbar)(\hbar\omega_k)]$ ;  $a_k^*(t) = a_k^* \exp[(it/\hbar)(\hbar\omega_k)]$  where  $a_k(t)$  and

<sup>(14)</sup> G. A. JONES and D. H. WILKINSON: *Phil. Mag.*, **44**, 542 (1953).

$a_k^*(t)$  are the operators for destruction and creation of positive mesons in the interaction representation and  $a_k, a_k^*$  the same in the Schrödinger representation; similar relations holding for negative and neutral mesons.

The integration over the virtual meson occupation numbers can in this way be done and we are left with a matrix element containing only quantities related to the nuclear system. We may write:

$$(A2) \quad \mathcal{M}' = \hbar^{-3} \sqrt{\frac{\hbar^2 c^2}{2f} \frac{c^2 g'^2}{V}} \frac{c}{2Mc} \sum_k \sum_{i,i'} \frac{\hbar^2}{2\omega_k} (\mathbf{k} \cdot \mathbf{e}_f) \langle \varphi'_2 | O_{ii'} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_{i'} | \varphi'_1 \rangle,$$

where we have put:

$$(A3) \quad O_{ii'} = \exp \left[ \frac{i}{\hbar} E_2 t \right] \cdot \exp [i \mathbf{k} \cdot \mathbf{x}_i] \cdot \exp \left[ -\frac{i}{\hbar} (H_N + \hbar \omega_k) t \right] \exp \left[ \frac{i}{\hbar} f t' \right] \cdot \\ \cdot \exp \left[ \frac{i}{\hbar} (\hbar \omega_k + H_N) t'' \right] \cdot \exp [-i \mathbf{k} \cdot \mathbf{x}_{i'}] \cdot \exp \left[ -\frac{i}{\hbar} E_1 t'' \right],$$

an operator depending on  $i, i'$ . In the neutral theory the only modification is to write 1 instead of  $\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_{i'}$ .

We have now to calculate:

$$(A4) \quad \gamma = \sum_{ii'} \langle \varphi'_2 | O_{ii'} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_{i'} | \varphi'_1 \rangle,$$

where  $\varphi'_1, \varphi'_2$  are now supposed to be given by expressions (28)<sub>1</sub> and (28)<sub>2</sub>.

We now make use of (30); we may thus perform the summation over the isotopic spin variables obtaining:

$$\gamma = \langle \varphi_2(\eta_1 \eta_2 \eta_3 \eta_4) | \sum_{ii'} O_{ii'} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_{i'} | \varphi_1(\eta_1 \eta_2 \eta_3 \eta_4) \rangle.$$

We shall now write:

$$\Phi_{1,2}(\eta_1 \eta_2 \eta_3 \eta_4) = \Phi_{1,2}(1, 2, 3, 4) \frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{\sqrt{2}} \frac{\alpha_3 \beta_4 - \alpha_4 \beta_3}{\sqrt{2}},$$

where  $\Phi$  is a function of the spatial coordinate only (in the following denoted simply by 1, 2, 3, 4).

We must at this point consider separately the various terms in the summation over  $i, i'$ . The terms with  $i = i'$  give together:

$$3 \langle \Phi_2(1234) | O_{11} + O_{22} + O_{33} + O_{44} | \Phi_1(1234) \rangle.$$

The terms with  $\begin{Bmatrix} i = 1, 2, 3, 4 \\ i' = 2, 1, 4, 3 \end{Bmatrix}$  give:

$$\langle \Phi_2(1234) | O_{12} + O_{21} + O_{34} + O_{43} | \Phi_1(1234) \rangle.$$

The remaining terms give:

$$\begin{aligned}
 & -\langle \Phi_2(1234) | O_{13} + O_{31} + O_{41} + O_{14} + O_{23} + O_{32} + O_{24} + O_{42} | \Phi_1(1234) \rangle, \\
 & -2 \langle \varphi_2(\eta_1 \eta_2 \eta_3 \eta_4) | (O_{13} + O_{31}) P_{13}^H + (O_{14} + O_{41}) I_{14}^H + \\
 & \quad + (O_{23} + O_{32}) P_{23}^H + (O_{24} + O_{42}) P_{24}^H | \varphi_1(\eta_1 \eta_2 \eta_3 \eta_4) \rangle.
 \end{aligned}$$

The summation over the spins has been done in all the preceding expressions except in the one of the last line; performing it we get:

$$\begin{aligned}
 & -\langle \Phi_2(1234) | (O_{13} + O_{31}) P_{13} + (O_{14} + O_{41}) P_{14} + \\
 & \quad + (O_{23} + O_{32}) P_{23} + (O_{24} + O_{42}) P_{24} | \Phi_1(1234) \rangle,
 \end{aligned}$$

where the  $P$  are now space exchange operators.

Extensive use will now be made of the symmetry properties of the  $\Phi$ . They are the following: *a*) they are symmetric with respect to the exchange of 1 with 2 or of 3 with 4 or both. *b*) They change sign if one exchanges 1 with 3 and 2 with 4 simultaneously. Making use of such properties  $\gamma$  can be written:

$$\begin{aligned}
 (A5) \quad \gamma &= 12 \langle \Phi_2(1234) | O_{11} | \Phi_1(1234) \rangle + 4 \langle \Phi_2(1234) | O_{12} | \Phi_1(1234) \rangle - \\
 & - 8 \langle \Phi_2(1234) | O_{13} | \Phi_1(1234) \rangle - 8 \langle \Phi_2(3214) | O_{13} | \Phi_1(1234) \rangle.
 \end{aligned}$$

The corresponding expression for the neutral theory is simply:

$$\sum_{ii'} \langle \Phi_2(1234) | O_{ii'} | \Phi_1(1234) \rangle.$$

We may at this point introduce a complete set of states  $\Phi_\alpha$  (energy  $E_\alpha$ ) for the nucleus.

Let's consider a typical term in  $\gamma$ , e.g.  $O_{12}$ ; introducing now explicitly the integrations with respect to the time variables, this assumes the form:

$$\begin{aligned}
 & \sum_{\alpha} \int_{-\infty}^{+\infty} dt \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \exp \left[ -\frac{i}{\hbar} (E_\alpha + \hbar \omega_k - E_2) t \right] \cdot \exp \left[ \frac{i}{\hbar} f t' \right] \cdot \exp \left[ \frac{i}{\hbar} (\hbar \omega_k - E_1 + E_\alpha) t'' \right] \cdot \\
 & \quad \cdot \langle \Phi_2(1234) | \exp [i \mathbf{k} \cdot \mathbf{x}_1] | \Phi_\alpha \rangle \langle \Phi_\alpha | \exp [-i \mathbf{k} \cdot \mathbf{x}_2] | \Phi_1(1234) \rangle,
 \end{aligned}$$

that is:

$$\begin{aligned}
 & 2\pi \hbar^3 \delta(E_1 - E_2 - f) \sum_{\alpha} \frac{1}{D_1^\alpha D_2^\alpha} \cdot \\
 & \quad \cdot \{ \langle \Phi_2(1234) | \exp [i \mathbf{k} \cdot \mathbf{x}_1] | \Phi_\alpha \rangle \langle \Phi_\alpha | \exp [-i \mathbf{k} \cdot \mathbf{x}_2] | \Phi_1(1234) \rangle \},
 \end{aligned}$$

where:

$$D_1^\alpha = E_1 - E_\alpha - \hbar \omega_k; \quad D_2^\alpha = E_2 - E_\alpha - \hbar \omega_k.$$

The part of the matrix element corresponding to the  $O_{12}$  term is then, remembering (A2):

$${}^{(12)}M' = \sqrt{\frac{\hbar^2 c^2}{2f}} \frac{c^2 g'^2}{V} \frac{e}{2Mc} \sum_{\alpha} \sum_k \frac{\hbar^2}{2\omega_k} (\mathbf{e}_f \cdot \mathbf{k}) \frac{1}{D_1^{\alpha} D_2^{\alpha}} \{ \langle \Phi_2(1234) | \cos \mathbf{k} \cdot \boldsymbol{\xi}_1 | \Phi_{\alpha} \rangle \cdot \langle \Phi_{\alpha} | \sin \mathbf{k} \cdot \boldsymbol{\xi}_2 | \Phi_1(1234) \rangle - \langle \Phi_2(1234) | \sin \mathbf{k} \cdot \boldsymbol{\xi}_1 | \Phi_{\alpha} \rangle \langle \Phi_{\alpha} | \cos \mathbf{k} \cdot \boldsymbol{\xi}_2 | \Phi_1(1234) \rangle \}.$$

Similar terms  ${}^{(11)}M'$ ,  ${}^{(13)}M'$ ,  ${}^{(13)}M'_{\text{exch.}}$  corresponding to the other terms in (A5) may be easily written and the total matrix element is:

$$(A6) \quad M' = 12 {}^{(11)}M' + 4 {}^{(12)}M' - 8 {}^{(13)}M' - 8 {}^{(13)}M'_{\text{exch.}}.$$

The matrix element for the neutral case is:

$$M = \sum_{ii'} {}^{(ii')}M = 4 {}^{(11)}M + 12 {}^{(12)}M.$$

The approximation described in the text will now be made, of cutting the sum over  $\mathbf{k}$  at some value  $|\mathbf{k}|1/a = \mu/a$  at the same time developing  $\sin \mathbf{k} \cdot \boldsymbol{\xi}_1$ ,  $\cos \mathbf{k} \cdot \boldsymbol{\xi}_1$  in a series of  $\mathbf{k} \cdot \boldsymbol{\xi}_1$ . There is here a difference between the neutral and the 3 charged cases. In the first one the first non vanishing term is of order  $k$  while in the second it is of order  $k$  due to the presence of  ${}^{(13)}M_{\text{exch.}}$  in (A6).

In order to see this point let us begin by substituting in the preceding expressions for the  ${}^{(ii')}M$ ,  $\cos \mathbf{k} \cdot \boldsymbol{\xi}_i$  by 1; the development of  $\sin \mathbf{k} \cdot \boldsymbol{\xi}_i$  will be done later. We get in this way for  ${}^{(12)}M'$ :

$${}^{(12)}M' = \sqrt{\frac{\hbar^2 c^2}{2f}} \frac{c^2 g'^2}{V} \frac{e}{2Mc} \sum_k \frac{\hbar^2}{2\omega_k} (\mathbf{e}_f \cdot \mathbf{k}) \left\{ \frac{\langle \Phi_2(1234) | \sin \mathbf{k} \cdot \boldsymbol{\xi}_2 | \Phi_1(1234) \rangle}{D_1^{\alpha=2} D_2^{\alpha=2}} - \frac{\langle \Phi_2(1234) | \sin \mathbf{k} \cdot \boldsymbol{\xi}_1 | \Phi_1(1234) \rangle}{D_1^{\alpha=1} D_2^{\alpha=1}} \right\}.$$

By remarking that

$$\langle \Phi_2(1234) | \sin \mathbf{k} \cdot \boldsymbol{\xi}_2 | \Phi_1(1234) \rangle = \langle \Phi_2(1234) | \sin \mathbf{k} \cdot \boldsymbol{\xi}_1 | \Phi_1(1234) \rangle,$$

this expression may be written as:

$${}^{(12)}M' = \sqrt{\frac{\hbar^2 c^2}{2f}} \frac{c^2 g'^2}{V} \frac{e}{2Mc} 2AE \sum_k \frac{\hbar^2}{2\omega_k} \frac{(\mathbf{e}_f \cdot \mathbf{k})}{(\hbar\omega_k)^3} \langle \Phi_2(1234) | \sin \mathbf{k} \cdot \boldsymbol{\xi}_1 | \Phi_1(1234) \rangle,$$

where the approximation has been done of neglecting in the denominator  $(E_2 - E_1)^2$  as compared to  $(\hbar\omega_k)^2$ .



The matrix elements for the charged case follow as:

$$M' = 8 \sqrt{\frac{\hbar^2 c^2}{2f} \frac{e^2 g'^2}{V}} \frac{e}{2Mc} 2\Delta E \sum_k \frac{\hbar^2}{2\omega_k} \frac{(\mathbf{e}_r \cdot \mathbf{k})}{(\hbar\omega_k)^3} \{ \Phi_2(1234) | \sin \mathbf{k} \cdot \boldsymbol{\xi}_1 | \Phi_1(1234) \rangle - \\ - \langle \Phi_2(3214) | \sin \mathbf{k} \cdot \boldsymbol{\xi}_1 | \Phi_1(1234) \rangle \}.$$

Developing now  $\sin \mathbf{k} \cdot \boldsymbol{\xi}_1$  in series of  $k$  it is easy to realize that while  $\langle \Phi_2(1234) | \sin \mathbf{k} \cdot \boldsymbol{\xi}_1 | \Phi_1(1234) \rangle$  vanishes at the first order  $\langle \Phi_2(3214) | \sin \mathbf{k} \cdot \boldsymbol{\xi}_1 | \Phi_1(1234) \rangle$  does not. We have, more precisely:

$$\langle \Phi_2(1234) | \sin \mathbf{k} \cdot \boldsymbol{\xi}_1 | \Phi_1(1234) \rangle \rightarrow -\frac{a^3}{16} k_z k^2,$$

$$\langle \Phi_2(3214) | \sin \mathbf{k} \cdot \boldsymbol{\xi}_1 | \Phi_1(1234) \rangle \rightarrow \frac{1}{2} k_z a,$$

leading to the expressions already given in the text.

## RIASSUNTO

Viene rilevato come ci sia da aspettarsi un effetto mesonico di indebolimento su una nota regola di selezione (per radiazioni di dipolo tra livelli con lo stesso spin isotopico in nuclei selfconiugati). L'effetto è dovuto al fatto che l'impulso totale dei nucleoni e del campo mesonico in un nucleo è una costante del moto, ma non l'impulso totale dei soli nucleoni; è tuttavia con quest'ultima ipotesi che la regola di selezione anzidetta viene di solito dedotta. Abbiamo calcolato il fattore di indebolimento che risulta essere (in ampiezza) dell'ordine  $(g^2/4\pi)(\mu/Mc^2) \cong U/Mc^2$ , dove  $\mu$  è l'inverso della lunghezza d'onda Compton del mesone,  $M$  la massa del nucleone,  $U$  un qualche valore medio dell'energia potenziale nel nucleo. Si tratta dello stesso ordine di grandezza dell'effetto dovuto all'impurità isotopica coulombiana. Se risulterà possibile, col perfezionarsi delle misure, separare, per lo meno per qualche transizione, l'effetto mesonico da quello coulombiano, ciò costituirà una prova sperimentale diretta dell'esistenza di mesoni virtuali nel nucleo, o, in altre parole, di una teoria relativistica (mesonica) delle forze nucleari.

## Recent Results on S-Particles. (\*)

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**Summary.** — To the present time, 40 examples of S-particles have been photographed with the M.I.T. multiplate cloud chamber. Examination of the pictures has yielded the following results. In three cases the charged secondary particles were found to have ranges of the order of  $100 \text{ g cm}^{-2}$  of lead. In three additional cases the secondary particles traversed more than  $70 \text{ g cm}^{-2}$  of lead before leaving the chamber. Shorter ranges were also found but, in interpreting these observations, one must consider the possibility of anomalous energy losses by nuclear collisions. In 9 cases an electron cascade appears to be associated with the decay of an S-particle. It is possible to interpret these events as two-body decay processes into a neutral and a charged  $\pi$ -meson. Such a decay scheme, however, cannot explain all observed S-events.

This note summarizes the results on S-particles obtained to the present time with the M.I.T. multiplate cloud chamber at Echo Lake, Colorado <sup>(1-5)</sup>. By S-particles we understand charged particles heavier than  $\pi$ -mesons, that

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(+) National Science Foundation Predoctoral Fellow.

<sup>(1)</sup> H. S. BRIDGE and M. ANNIS: *Phys. Rev.*, **82**, 445 (1951).

<sup>(2)</sup> B. ROSSI, H. S. BRIDGE and M. ANNIS: *Rend. Accad. Naz. Lincei*, **11**, 73 (1951).

<sup>(3)</sup> M. ANNIS, H. S. BRIDGE, H. COURANT, S. OLBERT and B. ROSSI: *Nuovo Cimento*, **9**, 624 (1952).

<sup>(4)</sup> H. S. BRIDGE, CH. PEYROU, B. ROSSI and R. SAFFORD: *Phys. Rev.*, **90**, 921 (1953).

<sup>(5)</sup> H. S. BRIDGE, H. COURANT, H. C. DE STAEBLER and B. ROSSI: *Phys. Rev.*, **91**, 1924 (1953).

appear to decay at rest, each producing a single fast charged secondary particle, assumed to be either a  $\mu^-$  or  $\pi^-$  meson.

In all of the experiments here considered, the chamber contained either lead plates, 0.25 inches thick or brass plates, 0.50 inches thick. The brass used for the plates had a density of  $8.72 \text{ g cm}^{-3}$ ; its composition by weight was 85% copper and 15% zinc.

We have observed altogether 40 examples of S-particles which, for the sake of analysis, we divide into two groups. The first group includes 19 events in which the secondary charged particle leaves the chamber after traversing less than  $60 \text{ g cm}^{-2}$  of lead equivalent. Since the particles of this group are identified as S-particles mainly from the behavior of the charged secondary particle, some criterion must be adopted to ensure that events arising from other known decay processes are not included in the data. For an event to be included in Group I, we have therefore required that the charged secondary product traverse at least  $15 \text{ g cm}^{-2}$  of lead at minimum ionization. This rules out decay processes such as  $\pi^\pm \rightarrow \mu^\pm + \nu$  and  $\tau^\pm \rightarrow 2\pi^\pm + \pi^\mp$  or  $\tau^\pm \rightarrow \pi^\pm + 2\pi^0$ , and makes the detection of decay processes such as  $\mu^\pm \rightarrow e^\pm + 2\nu$  extremely unlikely. The total amount of material penetrated by all secondary particles of the S-events included in the first group is  $562 \text{ g cm}^{-2}$  of lead plus  $98 \text{ g cm}^{-2}$  of brass.

The second group includes 21 events which we consider as particularly significant for one or more of the following reasons:

- (a) The charged decay product leaves the chamber after traversing more than  $60 \text{ g cm}^{-2}$  of lead.
- (b) The charged decay product stops in the chamber.
- (c) There is an electron cascade associated with the event.

The total amount of material penetrated by all secondary particles of the S-events included in the second group is  $720 \text{ g cm}^{-2}$  of lead plus  $384 \text{ g cm}^{-2}$  of brass. Details of the events in the second group are given in Tables I and II.

Table I lists the quantities of interest for events included in Groups (a) and (b) above. In Column 3 we give the observed penetration of the charged secondary particle, i.e., the amount of material (of the kind indicated in Column 2) traversed by this particle from its point of origin to the last visible section of the trajectory. This is a lower limit for the range of the particle. In cases where the secondary particle came to rest before leaving the cloud chamber, an upper limit to the range is also given. This was derived under the assumption that the secondary particle just failed to emerge from the plate in which it appeared to stop. Column 4 lists the range limits converted to lead equivalent. This conversion is in doubt to the extent of the uncert-

TABLE I. — *Summary of the data for S-events included in groups (a) and (b).*

Event number	Plate material	Range limits (from observed penetration) (g cm <sup>-2</sup> )	Range limits in lead equivalent (g cm <sup>-2</sup> )	Estimated limits of relative specific ionization of last segment of secondary track	Range limits in lead equivalent from ionization estimate (*) (g cm <sup>-2</sup> )	Remarks
H-1	Lead	19-29.5	—	< 3	23-29.5	Secondary stops Electron cascade
TR-2	Lead	> 92	—	—	—	—
12 668 (S3)	Lead	64.4-74.3	—	—	—	Secondary may not stop
26 110 (S4)	Lead	53-67	—	> 4	53-60	Secondary stops Electron cascade
26 653	Lead	> 73.6	—	—	—	—
39 567 (S7)	Lead	> 60	—	—	—	—
58 649	Lead	> 66	—	—	—	—
65 479	Lead	> 64.3	—	—	—	—
77 039	Brass	> 52	> 68	—	—	—
78 003	Brass	> 49	> 64	—	—	—
81 558	Brass	75.8-107.2	98.4-140	< 4	100-140	Secondary stops
85 380	Brass	67.7-81.7	88.7-108	< 3	92-108	Secondary stops
85 415	Brass	> 63.2	> 82.8	—	—	—
92 329	Brass	> 75.7	> 99.3	2½-5	100-104	—

(\*) These limits do not include the statistical error in estimating the mean range. As pointed out in the text, this error amounts to about 2.8%. Also, except in the case of 26 110, the limits do not include errors in reconstruction which amount to about 1% in range.

ainties in the range-energy relation. We have used the range-energy relations given by ARON <sup>(6)</sup>.

In some cases it is possible to estimate the specific ionization of the track at the point where it is last seen. In these cases one can set some limits to the residual range of the particle when it enters the last plate and, therefore, assign to the total range of the particle narrower limits of error than those obtained from the observed penetration. These limits are listed in Column 6. We have taken into consideration the inclination of the track, and we have assigned conservative limits to the estimated ionization (Column 5). In the computations we have assumed that the secondary particle is a  $\mu$ -meson. The results would be about one percent higher if the particle were a  $\pi$ -meson.

<sup>(6)</sup> W. A. ARON: *University of California, Thesis* (1951), (UCRL 1325) unpublished.



The relative standard error in the measured penetration which arises from errors in the reconstruction of the trajectory is in general less than 1%. The error in locating the point of decay is in general less than  $\pm 0.5$  mm; this corresponds to an error in the penetration of  $\pm 0.6$  g cm<sup>-2</sup> of lead.

In computing the momentum or energy from an observed range, one should bear in mind the statistical fluctuations in the ionization loss. For a given particle and material, the *mean* range is a unique function of the momentum or energy. However, the relative straggling in range of a  $\pi$ -meson or  $\mu$ -meson whose mean range is between 65 and 500 g cm<sup>-2</sup> of lead amounts to about 2.8% (?). The corresponding error in momentum is about half as great. The uncertainty in the range-energy relation mentioned above also affects the quoted values of the momenta to the extent of a few percent.

With regard to the charged secondary products, we remark the following:

(1) *There are 5 cases in which the secondary particle appears to stop in the chamber:*

Event no. 12668. For this event the chamber was not operating satisfactorily. On this account one cannot be certain whether or not the secondary particle penetrated the plate in which it seems to stop. If the particle stops, the range is between 64.4 and 74.3 g cm<sup>-2</sup> of lead.

Event no. 26110. The secondary particle is heavily ionizing when it enters the plate in which it stops. The range in g cm<sup>-2</sup> of lead is  $(53 - 60) \pm 2.8\%$ . In this event an electron cascade is apparently associated with the decay (\*).

Event H-1. The primary enters from the top, penetrates 6 plates, and stops in plate no. 7. The secondary particle is emitted downward; it penetrates one plate and stops in plate no. 9. It enters the last plate apparently at minimum ionization. Taking into account the ionization estimate, the range of the secondary in g cm<sup>-2</sup> of lead is between the limits  $(23 - 29.5) \pm 2.8\%$ . The stopping could be the result of collision loss, or it is also possible that the particle is a  $\pi$ -meson undergoing catastrophic absorption. This is one of the events in which an electron cascade is associated with the decay.

(?) K. R. SYMON: *Harvard University, Thesis* (1948) as quoted in B. ROSSI: *High-Energy Particle* (New York, 1952).

(\*) *Note added in proof.* — In view of the increasing importance attached to range measurements we have recently reconsidered our analysis methods in detail and have reanalyzed all our previous data. Event 26110 is the only case in which the redetermination of the range appreciably affects previously published results. (The range limits for this case were 65.6-67.7 g cm<sup>-2</sup> of lead.) In this event the range determination is inherently of lower accuracy than in most other cases, largely because of the obliqueness of the track. As a result of the new determination the ranges of 26110 and 12668 no longer overlap. Thus the previous argument in favour of a unique range group based on these two events is invalid.

Event no. 81558. The primary particle enters at the top, penetrates three plats and stops in plate No. 8. The limits to the range in  $\text{g cm}^{-2}$  of lead equivalent are  $(100 - 140) \pm 2.8\%$ . It is worth remarking that  $I/I_{\min}$  is about two for the last visible section of the trajectory. Thus if the last track section was caused by an electron which arose from  $\mu$ -e or  $\pi$ - $\mu$ -e decay of the secondary meson in the preceding plate, it should scatter markedly in the gas because of the low momentum. Since this behaviour is not observed, it seems impossible to attribute the last track section to such a process.

Another possible, though very unlikely, explanation is that the event represents a chance coincidence. The track attributed to the primary S-particle could be that of a stopped proton, and the track attributed to the secondary decay product could be that of an unrelated meson.

Event no. 85380. The primary enters from the top, penetrates 10 plates and stops in plate no. 11. The charged secondary is emitted upward, penetrates 4 plates and stops in plate no. 6. The limits to the range in  $\text{g cm}^{-2}$  of lead equivalent are  $(92 - 108) \pm 2.8\%$ .

The range limits quoted in the above five cases were derived taking into account the estimated specific ionization where this procedure was applicable. See Table I.

(2) *There is one case in which the secondary is heavily ionizing when it leaves the chamber.*

Event no. 92329. The primary particle is produced in an interaction in plate no. 3. It stops in plate no. 5. The secondary is emitted downward and leaves the chamber after traversing 6 brass plates. The increase in ionization in the last visible space is unmistakable. Taking into account the upper and lower limits for the ionization in this space, the limits on the range are, for a  $\mu$ -meson,  $(100 - 104) \pm 2.8\%$   $\text{g cm}^{-2}$  of lead equivalent.

The last three events given above are consistent with a single secondary range of about  $102 \text{ g cm}^{-2}$  of lead. This corresponds to a momentum of 266 MeV/c for a  $\pi$ -meson and 236 MeV/c for a  $\mu$ -meson. From these values of the momentum one obtains the following values for the primary mass,  $m_s$  (computed under the assumption of a single neutral secondary of zero mass): if the charged secondary is a  $\pi$ -meson,  $m_s = 1110 m_e$ ; if the charged secondary is a  $\mu$ -meson,  $m_s = 969 m_e$ .

(3) *There are three other cases in which the secondary does not stop but its range exceeds  $70 \text{ g cm}^{-2}$  lead equivalent (TR-2, 26653 and 85415).*

(4) *Our data are not consistent with the assumption that all charged secondaries are  $\mu$ -mesons of a single energy, for there are at least two cases where the secondary range is shorter than most other ranges. However, the data*

are consistent with the assumption that all charged secondaries are  $\pi$ -mesons of a unique energy; energy losses by nuclear collisions would then account for the short-range cases. In this connection one should note the following. Nuclear collisions in metal plates by  $\pi$ -mesons in the momentum range considered here are very inconspicuous. They result mainly in apparent stopping or large-angle scattering. Nuclear stars appear in less than 1% of the cases <sup>(8)</sup>. In our experiments there are three cases which *can* be interpreted as nuclear stopping or scattering if the secondary particles are  $\pi$ -mesons.

The data are, of course, consistent also with the assumption that the charged secondaries are a mixture of  $\pi$ -mesons and  $\mu$ -mesons, or with the assumption that they are  $\mu$ -mesons of different energies. Under any of these assumptions, however, the energy spectrum of the secondary particles must be strongly peaked towards the high energy end. This conclusion follows from the fact that so few secondary particles have been observed to stop in the chamber and also from a statistical analysis of the observed distribution of scattering angles for the secondary. If one assumes that all charged secondaries are mesons of the same mass and momentum, then according as they are assumed to be  $\mu$ -mesons or  $\pi$ -mesons, the analysis leads to a most likely value of the range of  $115 \pm 16$  g cm<sup>-2</sup> and  $101 \pm 14$  g cm<sup>-2</sup> of lead respectively. The analysis was carried out for a group of secondary particles which did not stop in the chamber. Ionization losses were taken into account.

TABLE II. - Summary of the data for S-events included in group (c).

Event number	Plate material	Range limits (from observed penetration) (g cm <sup>-2</sup> of Pb)	Non-ionizing link?	Angle between cascade and charged secondary	No. of electrons	Remarks
H-1	Pb	19-29.5	Yes	180°	3	Secondary stops
02877	Pb	> 44.5	Yes	127°	3	—
03246	Pb	> 13.3	No	147°	4	—
25178	Pb	> 29	No	180°	5	—
26049	Pb	> 47	No	157°	8-9	—
26110	Pb	53-67	Yes	163°	3	Secondary stops
(S4)						
32531	Pb	~ 0	Yes	180°	5	—
64700	Pb	> 32	No	—	2	Angle indetermin.
64888	Pb	> 49.2	Yes	123°	4	—

Table II lists pertinent data for events in which an electron cascade is associated with the decay (Group c above). Column 4 indicates whether or

<sup>(8)</sup> J. O. KESSLER: *Columbia University, Thesis* (1953) unpublished.

not there was a non-ionizing link between the point at which the decay occurred and the origin of the electron shower. Column 5 gives the space angle between the direction of the charged secondary product and the apparent trajectory of the photon which produced the cascade. These angles have errors of  $\pm 2^\circ$  when there is a non-ionizing link and errors of  $\pm 10^\circ$  in cases where the shower begins in the same plate where the S-particle comes to rest. Column 6 gives the total number of electron tracks seen in the various sections of the chamber through which the shower develops.

With regard to the neutral secondary particles we remark the following:

(1) *The chance association between S-particle decays and some of the small electron showers frequently observed is, of course, a possibility that must be considered.* From the frequency of these showers in our S-particle pictures we can calculate the probability that an unrelated shower should appear to originate from the point of decay of an S-particle. This probability is about  $10^{-3}$ . Thus the probability of one or more chance associations in our group of 40 S-particles is 0.04. The probability that *all* nine of our cases be chance associations, that is, that not one of the «associated» cascades is actually due to a photon coming in some way from this decay, is  $(.001)^9 = 10^{-27}$ .

(2) *The electron cascades appear to be produced by photons,* for in several cases they originate in a plate separated by as much as two radiation lengths from the one in which the S-particle stops, and there is no visible track connecting the point of decay with the origin of the shower.

(3) *Our data are not consistent with the assumption of a two-body decay at rest into a photon and a charged particle.* In some cases the photon and the charged secondary particle are not emitted in opposite directions; also, the total number of photons that have been detected is much smaller than that expected under the above assumption.

(4) *The observed distribution of angles between the photon and the charged decay product does not contradict the assumption that all photons arise from the decay of  $\pi^0$ -mesons,* emitted in the direction directly opposite to that of the charged decay product, with a momentum of the order of 200 MeV/c. Nor is this assumption contradicted by the failure to observe the second photon which is produced by the decay of the  $\pi^0$ -meson. Indeed, in all cases the computed energy and trajectory of the second photon are such as to make its detection very unlikely.

(5) *The assumption that all S-events correspond to the same two-body decay process, giving a  $\pi^0$ -meson and a charged particle as secondaries, is extremely difficult to reconcile with the comparatively small number of cases in which a photon has been detected.* The probability of detecting 9 or fewer cascades in



our sample of 40 S-particles is about  $3 \cdot 10^{-3}$ . Thus, while the above decay scheme may explain the S-events associated with photons, it cannot explain all S-events included in our sample.

(6) *The assumption that all S-events correspond to the same three-body decay process, each giving rise, for example, to a photon, a neutrino and either a  $\pi^-$  or a  $\mu^-$ -meson cannot be ruled out.*

Some general points should be kept in mind in drawing conclusions from our data.

In our experiments the only methods of estimating the mass of the primary particle are those that use range-scattering or range-ionization criteria. These criteria do not usually allow us to distinguish between heavy mesons and hyperons in any individual case. It is possible that in a few of our events the primary mass could be heavier than that of a proton.

Our estimate of the residual range from the specific ionization of the last visible track section does not allow us to determine whether or not the primaries of our S-events stop in the plates before they decay. If the primary decays in slow flight, the measured range of the secondary can be quite different from the range in the rest frame of the primary. In our sample of S-particles there are 19 secondaries emitted forward and 21 secondaries emitted backward in the laboratory system. In the three cases in which the range of the secondary particle suggests a single range of about  $102 \text{ g cm}^{-2}$  of lead one (92399) is emitted forward, one (85380) is emitted backward and one (81558) is emitted at about  $90^\circ$ . These facts suggest that most of the S-particles come to rest before decaying.

$\tau$ -mesons undergoing the decay:

$$(1) \quad \tau^\pm \rightarrow 2\pi^\pm + \pi^\mp$$

or the alternate decay:

$$(2) \quad \tau^\pm \rightarrow \pi^\pm + 2\pi^0$$

give rise to secondary charged  $\pi$ -mesons with a maximum range of about  $20 \text{ g cm}^{-2}$  of lead. In our sample of S-particles, there is no event that can be interpreted according to the decay scheme (1) and only one event that can be interpreted according to the decay scheme (2). The latter is event No. 32531 in which the charged secondary leaves the illuminated region without traversing an appreciable amount of material, and in which an associated electron cascade appears.

One must consider the possibility that some of our S-events may be the result of the nuclear absorption of  $K^-$  mesons.

We would expect our detection and scanning efficiencies to be about the same for all secondaries with ranges greater than about  $7 \text{ g cm}^{-2}$  of lead or  $11 \text{ g cm}^{-2}$  of brass. This should be kept in mind in considering possible energy spectra for the secondary particles.

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### RIASSUNTO

Quaranta esempi di particelle S sono stati fotografati finora colla camera di Wilson a placche del Massachusetts Institute of Technology. L'esame delle fotografie ha dato i seguenti risultati. In 3 casi il percorso delle particelle cariche secondarie risulta essere dell'ordine di  $100 \text{ g cm}^{-2}$  di piombo. In 3 altri casi la particella secondaria traversa oltre  $70 \text{ g cm}^{-2}$  di piombo prima di uscire dalla camera. Talora il percorso è più breve, ma nell'interpretare queste osservazioni bisogna tener conto della possibilità di perdite d'energia per mezzo di interazioni nucleari. In 9 casi vi è uno sciame elettronico associato colla disintegrazione di una particella S. È possibile interpretare questi casi coll'ipotesi di una particella che si disintegra producendo un mesone  $\pi$  neutro ed un mesone  $\pi$  carico. Però, non tutte le particelle S da noi osservate possono essere di questo tipo.

## High Nuclear Polarizations in Paramagnetic Substances.

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**Summary.** — In this work the equilibrium of nuclear polarization in paramagnetic substances is studied from the statistical viewpoint. The polarization is the higher, the nearer the paramagnetic resonance of the electrons is to saturation. The results obtained do not differ substantially when the electrons behave as a Fermi gas or when the Boltzmann approximation is valid.

### Introduction.

In a recent work A. W. OVERHAUSER <sup>(1)</sup>, investigated the paramagnetic relaxation process concerning the conduction electrons in alkali metals. He showed that it is possible to obtain nuclear polarizations which are the higher, the nearer is the paramagnetic resonance of the electrons to saturation. This effect depends on the interaction between the spin moments of the electron and nucleus.

Every time that this interaction increases the nuclear spin moment projection along the magnetic field direction by one unity, it diminishes the electron spin moment projection by the same amount.

In this work it is shown that the equilibrium of the nuclear polarization does not depend on the relaxation times. Indeed it is only a function of the magnetic field, temperature and degree of saturation reached in the paramagnetic resonance. Furthermore, the results concerning the nuclear polarization in the case of electrons obeying Boltzman statistics are not very different from those obtained in the case of electrons behaving like totally degenerate Fermi gas. The conclusions of the present paper based on statistical

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<sup>(1)</sup> A. W. OVERHAUSER: *Phys. Rev.*, **92**, 411 (1953).

arguments coincide with those obtained by simple thermodynamical considerations <sup>(2)</sup>. In this work Fowler's statistical method will be used, which is founded upon the use of the great partition function  $\Xi$  allowing one to develop synthetically the necessary calculations <sup>(3)</sup>.

### Theory.

a) *Great partition function for a paramagnetic assembly.* — It is known that the general expression for the great partition function of an assembly of particles of a single type is <sup>(3)</sup>:

$$\Xi = \prod_s^{\infty} \left( 1 \pm \lambda \exp \left[ -\frac{\varepsilon_s}{kT} \right] \right)^{\pm \omega_s},$$

where  $\varepsilon_s$  are the energies accessible to the particles,  $\omega_s$  the corresponding statistical weights and  $\lambda$  is related to the chemical potential  $\mu$  through the equation:

$$(1) \quad \mu = kT \lg \lambda.$$

The upper sign means Fermi-Dirac statistics the lower one Bose-Einstein.  $I$  means the spin moment of the particles; the energies  $\varepsilon_s$  in presence of a magnetic field  $H$  are:

$$(2) \quad \varepsilon_s = -mg\beta H + \varepsilon_r^0, \quad (-I \leq m \leq I),$$

where  $m$  is the component of  $I$  in the direction of the field,  $g$  the gyromagnetic ratio,  $\beta$  the magneton associated with the particles and  $\varepsilon_r^0$  their energies in absence of a field. As in the presence of a magnetic field the energy levels associated with the spin coordinate are not degenerate, the statistical weight of the  $\varepsilon_s$  level will depend on the degeneracy degree related to the other coordinates of a particle; hence we shall indicate it by  $\omega_r$ . So far we have postulated a perfect behaviour of the system, that is the absence of any interaction associated with the spin moments. On these hypotheses the expression for  $\Xi$  is:

$$(3) \quad \Xi = \prod_{m=-I}^I \Xi_m,$$

<sup>(2)</sup> P. BROVETTO and G. CINI: *Nuovo Cimento*, **11**, 618 (1954).

<sup>(3)</sup> R. H. FOWLER: *Proc. Cam. Phil. Soc.*, **34**, 382 (1938); see also: R. H. FOWLER and E. A. GUGGENHEIM: *Statistical Thermodynamics* (Cambridge, 1949).



where:

$$(4) \quad \mathcal{E}_m = \prod_0^{\infty} [1 \pm \lambda_m \exp [mg\beta H - \varepsilon_r^0]]^{\pm \omega_r};$$

in (4)  $\lambda$  has been signed by an index  $m$ , but for a balanced sistem only one value of  $\lambda$  exists, that is:  $\lambda_m = \lambda_{m+1}$ , etc..

Considering (3) it is clear that in the presence of a magnetic field the system is equivalent to a mixture of  $2I+1$  types of different particles, each one associated with a function  $\mathcal{E}_m$ . For the sake of simplicity we shall always consider  $1 \text{ cm}^3$  of the substance. We call  $N$  the total number of particles and  $N_m$  the number of particles having component  $m$  of the spin in the direction of the magnetic field, that is to say:

$$(5) \quad N = \sum_{-I}^I N_m.$$

As is known, we have:

$$(6) \quad N_m = \lambda_m \left( \frac{\partial \lg \mathcal{E}}{\partial \lambda_m} \right)_{T, H}.$$

In (6) it is obviously essential to mark with an index  $m$  the  $\lambda$  as the derivation concerns but one of the factors appearing in the expression (3) for  $\mathcal{E}$ . For the case of a paramagnetic substance, the elementary work for a variation  $dH$  of the magnetic field is  $F_H dH$ ,  $F_H$  being the component of the magnetic moment in the direction of the field. The polarization is defined as follows:

$$P = \frac{F_H}{Ng\beta I}.$$

From known formulae we have then:

$$(7) \quad P = \frac{kT}{Ng\beta I} \left( \frac{\partial \lg \mathcal{E}}{\partial H} \right)_{\lambda_m, T}.$$

Our purpose is to evaluate the great partition function for an assembly of nuclei, assuming their interaction with electrons to be expressed by the Hamiltonian (4):

$$(8) \quad \frac{8\pi}{3} g_n g_e \beta_n \beta_e \sigma_n \sigma_e \delta(r),$$

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(4) E. FERMI: *Zeits. f. Phys.*, **60**, 320 (1930).

where the index  $n$  is related to nuclei, index  $e$  to electrons,  $\sigma$  are the spin operators and  $\delta(r)$  is Dirac's function of the distance between electron and nucleus. By (7) we get then immediately the expression for the nuclear polarization  $P$ . We observe that for the nuclei the classical Boltzmann approximation can always be used, that is  $\lambda_m \ll 1$ . In this case, putting

$$\sum_r^{\infty} \omega_r \exp \left[ -\frac{\varepsilon_r^0}{kT} \right] = f_n(T),$$

we have:

$$(9) \quad \lg \Xi_n = f_n(T) \sum_{-I}^I \lambda_m \exp \left[ \frac{mg_n \beta_n H}{kT} \right],$$

where  $f_n(T)$  is the small partition function of the nuclei. In the case of an alkali metal the conduction electrons behave as a completely degenerate Fermi gas, so that the classical approximation cannot be used. In this case the energies are continuously distributed, and the statistical weight of the levels comprised between  $\varepsilon_r$  and  $\varepsilon_r + d\varepsilon_r$  is:

$$(9) \quad \omega_r = 2 \frac{\pi}{4} \frac{(8 m_e)^{\frac{3}{2}}}{h^3} \varepsilon_r^{\frac{1}{2}} d\varepsilon_r.$$

Therefore, taking into account the fact that for the electrons  $g_e = 2$ , we have:

$$(10) \quad \lg \Xi_e = \frac{(2\pi m_e kT)^{\frac{3}{2}}}{h^3} \frac{2}{\sqrt{\pi}} \left\{ \int_0^{\infty} x^{\frac{1}{2}} \lg \left( 1 + \lambda^+ \exp \left[ -x + \frac{\beta_e H}{kT} \right] \right) dx + \right. \\ \left. + \int_0^{\infty} x^{\frac{1}{2}} \lg \left( 1 + \lambda^- \exp \left[ -x - \frac{\beta_e H}{kT} \right] \right) dx \right\}.$$

When it is possible to apply the Boltzmann approximation to the electrons, we have on the contrary:

$$(11) \quad \lg \Xi_e = f_e(T) \left( \lambda^+ \exp \left[ \frac{\beta_e H}{kT} \right] + \lambda^- \exp \left[ -\frac{\beta_e H}{kT} \right] \right);$$

in analogy to (9) which is valid for the nuclei, the (+) and (-) signs mean the two spin states of the electron.

b) *Relations between  $\lambda^+$  and  $\lambda^-$  for electrons under paramagnetic resonance.*

– When the electrons are strongly irradiated by microwaves having a frequency  $\nu = (2|\beta_e|H)/h$  corresponding to paramagnetic resonance, the transitions bet-

ween the spin levels modify the occupation number  $N^+$ ,  $N^-$ . We can show that under such conditions the parameters  $\lambda^+$ ,  $\lambda^-$  are not equal. We indicate by  $N_v^+$ ,  $N_v^-$ ;  $N_0^+$ ,  $N_0^-$  the values of the occupation numbers, in the presence and absence of microwaves respectively. We define according to OVERHAUSER a saturation parameter  $s$ :

$$(12) \quad s = 1 - \frac{N_v^- - N_v^+}{N_0^- - N_0^+},$$

obviously it increases from 0 to 1 as a function of the intensity of the field of the microwaves. Using (6) and (10) in the quantum treatment we have:

$$(13) \quad N^+ = \frac{(2\pi m_e kT)^{\frac{3}{2}}}{h^3} \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{x^2 dx}{1 + (1/\lambda^+) \exp [x - \beta_e H/kT]},$$

the expression for  $N^-$  is given by an analogous formula. The integral appearing in (13) can easily be evaluated observing that for  $x \ll \lg \lambda^+ + \beta_e H/kT$  the integrand is really  $x^2$ , while for  $x \gg \lg \lambda^+ + \beta_e H/kT$  it gets very small. Taking this into account we have:

$$(14) \quad \begin{cases} N^+ = \frac{(2\pi m_e kT)^{\frac{3}{2}}}{h^3} \frac{2}{\sqrt{\pi}} \frac{2}{3} \left( \lg \lambda^+ + \frac{\beta_e H}{kT} \right), \\ N^- = \frac{(2\pi m_e kT)^{\frac{3}{2}}}{h^3} \frac{2}{\sqrt{\pi}} \frac{2}{3} \left( \lg \lambda^- - \frac{\beta_e H}{kT} \right). \end{cases}$$

The formulae (14) represent, as we know, specially for low temperatures, an excellent approximation, which is more than the requirements of the theory demand. As in the absence of microwaves, the distribution parameters coincide, using formulae (12) and (14), and considering that  $-\beta_e = |\beta_e|$  we get:

$$(15) \quad \lg \frac{\lambda^+}{\lambda^-} = \frac{2s |\beta_e| H}{kT},$$

from which it follows that for  $s \neq 0$  we have  $\lambda^- \neq \lambda^+$ . Under such conditions the electrons having a spin respectively parallel and antiparallel to the magnetic field are not in equilibrium with each other, in fact they are in mutual equilibrium with the photons of the microwaves. In Boltzmann's approximation by means of the formulae (6) and (10) we have:

$$(16) \quad N^+ = \lambda^+ f_e(T); \quad N^- = \lambda^- f_e(T).$$

Using formulae (12) and (16) we obtain:

$$(17) \quad \lg \frac{\lambda^+}{\lambda^-} = 2 \frac{|\beta_e| H}{kT} - 2 \operatorname{arctgh} \left[ (1-s) \operatorname{tgh} \frac{|\beta_e| H}{kT} \right].$$

It immediately follows that for  $s = 0$  and  $s = 1$  formulae (15) and (17) coincide. Furthermore, for not too low temperatures formula (17) becomes practically the same as (15), as is easily understood remembering that the functions  $\operatorname{tgh}$  and  $\operatorname{arctgh}$  for small enough values of the arguments may be essentially identified with the argument.

c) *Formulae of nuclear polarization.* — Considering the interaction of the electrons with the nuclei by means of the Hamiltonian (8), it is easy to show that in the presence of microwaves the distribution parameters  $\lambda_m$  of the nuclei are also not equal to each other.

Each infinitesimal variation of the occupation numbers  $N^+$ ,  $N^-$ ,  $N_m$  when the equilibrium is reached must leave the free energy  $F$  of the system unchanged.

We must however remember that the occupation numbers are not completely independent; they are connected by some relations. From the equation (5) we have in fact:

$$(18) \quad N^+ + N^- - N_e = 0,$$

$$(19) \quad \sum_m^I N_m - N_n = 0,$$

where  $N_e$ ,  $N_n$  are the total number of electrons and nuclei per  $\text{cm}^3$  respectively. Moreover, the sum of the components in the direction of the magnetic field, of the spin operators of the electrons and nuclei commutes with the Hamiltonian of the system of all particles, including the interaction term (8). This means that the sum of such components must be conserved. Let us denote by  $S_H$  the component of the total spin moment of the system along the magnetic field; the following relation must then be true:

$$(20) \quad \frac{1}{2}N^+ - \frac{1}{2}N^- + \sum_m^I mN_m - S_H = 0.$$

Therefore the equilibrium state is determined by the equation:

$$dF(N^+, N^-, N_m) = 0$$

and by the conditions (18), (19), (20). Using three Lagrangian multipliers



$\alpha$ ,  $\beta$ ,  $\gamma$  for the conditions (18), (19), (20) we have then:

$$\frac{\partial}{\partial N^+} \left[ F(N^+, N^-, N_m) + \alpha (N^+ + N^- - N_e) + \beta \left( \sum_m N_m - N_n \right) + \right. \\ \left. + \gamma \left( \frac{1}{2} N^+ - \frac{1}{2} N^- + \sum_m m N_m - S_H \right) \right] = 0$$

analogous equations hold for  $N^-$ ,  $N_m$ . Performing the differentiations and introducing the chemical potential  $\mu = \partial F / \partial N$  these equations become:

$$\mu^+ + \frac{1}{2}\gamma + \alpha = 0$$

$$\mu^- - \frac{1}{2}\gamma + \alpha = 0$$

$$\mu_m + m\gamma - \beta = 0$$

from which, eliminating the multipliers  $\alpha$ ,  $\beta$ ,  $\gamma$  we obtain immediately:

$$\mu_+ - \mu_- - \mu_m + \mu_{m-1} = 0.$$

Using the equation (1), it ensues:

$$(21) \quad \frac{\lambda^+}{\lambda^-} = \frac{\lambda_m}{\lambda_{m-1}}.$$

Let us consider now the great partition function for the nuclei which are in equilibrium with the electrons under paramagnetic resonance. Taking into account (6), (9) and (5), we get:

$$(22) \quad N = f_n(T) \sum_m \lambda_m \exp \left[ \frac{mg_n \beta_n}{kT} H \right].$$

Using (15) and (17), (21) becomes in the quantum and classical case respectively:

$$(23) \quad \frac{\lambda_m}{\lambda_{m-1}} = \exp \left[ \frac{2s |\beta_e|}{kT} H \right],$$

$$(24) \quad \frac{\lambda_m}{\lambda_{m-1}} = \exp \left\{ 2 \frac{|\beta_e|}{kT} H - 2 \operatorname{arctgh} \left[ (1-s) \operatorname{tgh} \frac{|\beta_e| H}{kT} \right] \right\}.$$

The equations (22), (23) and (24) determine completely the parameters  $\lambda_m$  and then the expression (9) for the great partition function  $\Xi_n$  of the nuclei. Therefore we can formulate by (7) the expression for the nuclear polarization;

we see very soon by easy calculations that it is given by Brillouin's function:

$$(25) \quad P = \frac{2I+1}{2I} \coth (2I+1)\xi - \frac{1}{2I} \coth \xi,$$

where for the quantum case:

$$(26) \quad \xi = \frac{g_n \beta_n + 2s |\beta_e|}{2kT} H$$

and for the classical one:

$$(27) \quad \xi = \frac{g_n \beta_n + 2 |\beta_e|}{2kT} H - \operatorname{arctgh} \left[ (1-s) \operatorname{tgh} \frac{|\beta_e|}{kT} H \right].$$

### Conclusion.

In Fig. 1 the graphs of the nuclear polarization are drawn for the values  $4.28 \cdot 10^4$  and  $1.07 \cdot 10^4$  for  $H/T$  (in gauss/degree). The  $C$  curves illustrate the classical case, the  $Q$  curves the quantum one. From the graph it appears clearly that the values of polarization for  $H/T = 1.07 \cdot 10^4$  in the classical and quantum treatments are very similar. On the contrary, for  $H/T = 4.28 \cdot 10^4$  the values for the classical curve are much higher than for the quantum one. Anyway, in every case, for values of  $s$  near 1 they are practically tangent. The quantum statistical treatment in which the electrons behave like a completely degenerate Fermi gas refers, as we have observed to the alkali metals. The classical treatment, instead, describes the effect of nuclear polarization in those paramagnetic salts in which the electrons are found in bound states and obey Boltzmann statistics to a good approximation even at low temperatures. We can therefore conclude that when the value of  $H/T$  is sufficiently high, the value of the saturation parameters  $s$  required to obtain a polarization near unity is substantially lower in the non conducting paramagnetic salts than in the alkali metals. It is likely that the

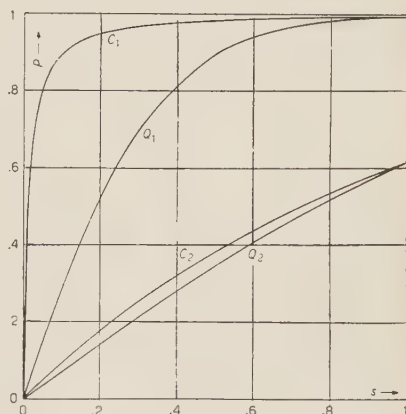


Fig. 1. — Values of  $P$  plotted against  $s$ . The curves  $C$  and  $Q$  are calculated by means of classical and quantum treatment respectively, the index 1 refers to  $H/T = 4.28 \cdot 10^4$  and the index 2 to  $H/T = 1.07 \cdot 10^4$ .

non alkali metals, in which the effect of the lattice electric field is important, behave in an intermediate manner but at the same time very similarly to the alkali metals. From the preceding statistical treatment it follows, anyhow, that the polarization effect is possible for all paramagnetic substances and not only in the case of alkali metals. On the contrary, considering the subject only from the statistical viewpoint nothing can be said about the time necessary for the equilibrium to be reached.

In foregoing calculations we assumed that the energies associated with the spin states of electrons and nuclei depend only on the magnetic field  $H$ ; indeed many interactions depending on the spin moments of the particles would intervene to complicate the problem. Anyway, we can abstract from them for studying the behaviour of the system in case of high values of nuclear polarization; in fact in this case they may be neglected in comparison to the high values necessary for the magnetic field.

The possibility of obtaining high nuclear polarization is obviously of great experimental interest. In a metal in which the nuclei are of the same type, it is of particular utility for scattering experiments, in non metals for the study of angular distributions in radioactive decay, on account of the fact that the presence of different nuclei cannot be an obstacle in this case.

### Aknowledgement.

We should like to express our thanks to Prof. G. WATAGHIN for useful discussions and criticism.

### RIASSUNTO (\*)

Nel presente lavoro si studia dal punto di vista statistico l'equilibrio della polarizzazione nucleare nelle sostanze paramagnetiche. La polarizzazione cresce all'approssimarsi della risonanza paramagnetica degli elettroni alla saturazione. I risultati ottenuti non differiscono sostanzialmente sia che gli elettroni si comportino come un gas di Fermi o che sia valida l'approssimazione di Boltzmann.

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(\*) Traduzione a cura della Redazione.

## Una precisazione sulle “tracce $T$ ” in emulsione nucleare.

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(ricevuto il 15 Maggio 1954)

**Riassunto.** — Esaminando i decadimenti  $\mu \rightarrow e$ , è stato determinato il limite superiore  $R_0$  del valore da assegnare alla distanza  $r$  tra una traccia al minimo e la fine di una traccia nera terminante in emulsione, nel caso che la associazione tra le due tracce non sia casuale. La distribuzione di queste distanze (fig. 1a) mostra che esse, nel 90% dei casi, sono contenute entro l'intervallo  $0 - 0,6 \mu$ . Ciò posto, si confronta, nell'intervallo  $0 - R_0$ , la distribuzione delle distanze  $r$  ricavate nel caso delle così dette « tracce  $T$  » con quella degli attraversamenti casuali, tra una traccia nera ed una al minimo, ottenuta sperimentalmente. Il confronto tra le due distribuzioni (fig. 1b) non mostra alcuna evidenza dell'esistenza di « tracce  $T$  » non casuali.

1. — BLAU e SALANT <sup>(1)</sup>, durante l'esplorazione di lastre nucleari Eastman NTB3 esposte per 5 ore a 28000 m, hanno osservato degli eventi, denominati « tracce  $T$  », formate da una traccia nera terminante in emulsione attraversata alla fine del suo percorso da un'altra al minimo di ionizzazione ed abbastanza diritta da escludere che si tratti di un elettrone di fondo o di bassa energia. Essi hanno notato che il numero di tali eventi è molto superiore (11 invece di 0,9) a quello dovuto a coincidenze casuali fra la fine di una traccia nera ed una al minimo ed avanzano l'ipotesi che tali eventi siano dovuti a una particella pesante che alla fine del suo percorso dà origine a due particelle veloci cariche.

BARRETT e coll. <sup>(2)</sup>, ripetendo l'osservazione su lastre G5 esposte per 10 ore ad un'altezza un po' inferiore ed alla stessa latitudine dei precedenti, trovano

<sup>(1)</sup> M. BLAU e E. O. SALANT: *Phys. Rev.*, **88**, 954 (1952).

<sup>(2)</sup> P. H. BARRETT, G. COCCONI e Y. EISENBERG: *Phys. Rev.*, **89**, 656 (1953).



invece che il numero di « tracce  $T$  » trovate concorda perfettamente con quello che ci si dovrebbe aspettare se tali eventi fossero tutti dovuti a coincidenze casuali.

La netta discordanza fra i risultati, se si escludono effetti molto sensibili dovuti alla quota o al materiale circostante le lastre, può essere dovuta sia alla valutazione più o meno sovrabbondante delle coincidenze casuali, sia al diverso criterio di scelta delle tracce stesse. Infatti mentre i primi includono nella statistica solo quelle « tracce  $T$  » in cui la distanza  $r$  fra il centro dell'ultimo grano della traccia nera e la linea passante per il baricentro della traccia al minimo non supera  $0,25 \mu$ , i secondi, tenendo conto delle difficoltà che per ragioni di profondità di fuoco si incontrano nel misurare distanze verticali inferiori ad  $1 \mu$ , confrontano con le coincidenze casuali, le « tracce  $T$  » in cui tale distanza è compresa in ognuno degli intervalli  $0 \vdash 1$ ,  $1 \vdash 2$ , ...,  $4 \vdash 5 \mu$ .

Per altro è evidente che la scelta del limite superiore da assegnare ad  $r$  è di particolare importanza, non solo per il fatto che da una certa distanza in poi tutte le « tracce  $T$  » sono da considerarsi sicuramente come attraversamenti casuali, ma anche perchè soprattutto per  $r$  molto piccoli, l'incertezza dovuta alla profondità di fuoco e all'errore nella misura di  $r$ , rendono il calcolo delle coincidenze casuali piuttosto incerto.

Per liberarci da queste difficoltà ci siamo proposti di determinare sia il limite superiore da assegnare ad  $r$ , sia il numero delle coincidenze casuali per via esclusivamente sperimentale.

2. — La determinazione del limite superiore da assegnare ad  $r$  è stata fatta riferendoci alla distribuzione di queste distanze nel caso di una coincidenza, sicuramente non casuale, tra una traccia nera e una al minimo, quale quella dovuta al decadimento  $\mu \rightarrow e$ , prendendo in considerazione quei decadimenti in cui l'elettrone era sufficientemente diritto e poco inclinato. Le misure sono state eseguite con ingrandimento  $1300\times$ , adoperando un oculare micrometrico che permetteva di misurare distanze orizzontali con un errore medio di  $\pm 0,1 \mu$ . Abbiamo trovato più conveniente misurare la distanza  $r$  tra la linea passante per il baricentro dei grani della traccia al minimo ed il punto più vicino della fine della traccia nera considerata, in quanto quest'ultimo punto poteva essere più facilmente individuato che non il centro del grano.

I risultati, relativi a 102 decadimenti  $\mu \rightarrow e$  trovati su emulsioni diversamente esposte, sono riportati in fig. 1a (\*): dal grafico risulta che tutti questi eventi hanno distanze  $r$  comprese nell'intervallo  $0 \vdash 1 \mu$  e che la loro distribuzione è tale che il 90% di essi è contenuto nell'intervallo  $0 \vdash 0,6 \mu$

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(\*) È stato possibile sommare i risultati delle diverse emulsioni essendo confrontabili tra loro i valori dei diametri medi dei grani.

e questo, sia quando si accettati per l'elettrone qualsiasi direzione di emissione (istogramma A), sia quando si escludono quei casi ove questa direzione forma con quella finale del mesone un angolo non superiore a  $30^\circ$  (istogramma B). Questa ultima considerazione è stata fatta perchè nel caso delle « tracce  $T$  » abbiamo preso in esame solo quelle ove l'angolo fra la direzione della traccia nera e quella della traccia al minimo non era inferiore al valore suddetto in quanto, per angoli più piccoli, la determinazione della distanza  $r$  era molto incerta. Si può pertanto concludere che se esistono eventi  $T$  che non siano casuali, essi dovranno nel 90% dei casi avere distanze  $r$  comprese fra  $0 \rightarrow 0,6 \mu$ .

3. — Ciò stabilito, nelle medesime lastre dove sono state eseguite le misure precedenti, è stata esaminata la fine di 6015 tracce nere, sia isolate che provenienti da stelle di evaporazione nucleare: i risultati sono riportati nella tabella seguente:

Emulsione	Esposizione	Superficie esplorata	N. tracce	N. $T$	N. casuali
G5 400 $\mu$	2h a 19000 m	27 cm <sup>2</sup>	1213 ( 502) (*)	3	2,7
G5 600 $\mu$	1h a 24000 m	19 cm <sup>2</sup>	1100 ( 350)	3	3,3
G5 600 $\mu$	5h a 22000 m	24,3 cm <sup>2</sup>	3702 (1142)	13	11,3

(\*) I numeri entro parentesi si riferiscono a quelli delle tracce da evaporazione nucleare.

In essa nella prima e seconda colonna sono riportate il tipo di emulsione (\*) e le condizioni di esposizione, nella terza la superficie esplorata, nella quarta il numero di tracce nere esaminate, nella quinta il numero di « tracce  $T$  » trovato, in cui  $r$  era compreso nell'intervallo  $0 \rightarrow 0,9 \mu$ , e nella sesta colonna il numero delle coincidenze casuali determinato sperimentalmente nel modo seguente.

Per ogni emulsione sono stati esaminati un numero di campi scelti a caso, eguale a quello delle tracce nere esaminate nella lastra medesima, e misurate, nelle stesse condizioni sperimentali delle « tracce  $T$  », le distanze  $r$  fra il centro del reticolo e la direzione della traccia al minimo passante casualmente nelle vicinanze di detto centro.

Si è così trovato il numero delle coincidenze casuali relativo a tracce al minimo formanti, rispetto ad una generica direzione, un angolo qualsiasi; tale

(\*) Delle emulsioni utilizzate in questa ricerca, la prima faceva parte di un lancio di palloni sonda, organizzato dal Centre de Physique Nucléaire di Bruxelles e dall'Istituto di Fisica di Roma, che ringraziamo vivamente, e le altre due del primo volo di Cagliari.

numero è stato ridotto di  $1/3$  per tener conto del fatto che nel caso delle « tracce  $T$  » da noi prese in esame le tracce al minimo formavano con la direzione della traccia nera un angolo non inferiore a  $30^\circ$ .

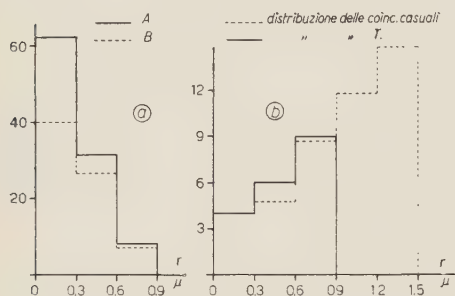


Fig. 1.

La distribuzione di dette coincidenze sino a distanze  $r = 1,5 \mu$ , è confrontata con quella delle « tracce  $T$  » sperimentali nel grafico di fig. 1b; in esso sono stati sommati i risultati delle diverse emulsioni in quanto nessun effetto dovuto sia alle condizioni di esposizione sia alla diversità di sviluppo è stato riscontrato.

Il perfetto accordo fra le due distribuzioni, nell'intervallo di distanze  $r$  ove le « tracce  $T$  » potrebbero anche essere degli eventi fisici, ci porta a concludere che nei limiti della precisione delle nostre misure non esiste alcuna indicazione sulla reale esistenza di eventi del tipo di quelli proposti da BLAU e SALANT.

Ringraziamo vivamente il prof. PANCINI per l'indirizzo, l'interessamento ed i suggerimenti datici ed il prof. BRAMBILLA per le utili discussioni.

## SUMMARY

To clear up the problem of the so-called «  $T$  tracks », it is interesting to know the maximum value  $R_0$  of the distances  $r$  between a black track ending in the emulsion and a minimum track, when the association between the two tracks is a surely established physical event. This maximum value was determined in the case of  $\mu \rightarrow e$  decays and it was, found that 90% of the distances  $r$  are contained in the range  $0 \rightarrow 0.6 \mu$  (Fig. 1a). The distribution of the distances  $r$  in the case of the «  $T$  tracks » was compared in the range  $0 \rightarrow R_0$  with the experimentally determined distribution of random coincidences. The close similarity of the two distributions (Fig. 1b) allow us to conclude that no evidence exists of the occurrence of non casual «  $T$  tracks ».

## On the Hydrodynamical Model of the Quantum Mechanics.

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(ricevuto il 17 Maggio 1954)

**Summary.** — A new hydrodynamical model for the Schrödinger equation is discussed. The new model differs from that of MADELUNG by the existence of turbulence. It follows directly from the ordinary interpretation of the quantum mechanics, by the introduction of operators for the charge and current densities and the components of the stress tensor in the one particle formalism. The model is developed for any values of the spin. The Madelung fluid corresponds to the mean motion of the special turbulent medium. The quantum potential appears as a combination of a pressure with terms arising from the turbulence. It is shown that the quantization of the motion of the Madelung fluid introduces the right kind of turbulence. The trajectories of the de Broglie-Bohm theory appear as trajectories of the mean motion of the turbulent medium.

### 1. — Introduction.

MADELUNG <sup>(1)</sup> showed that a hydrodynamical model can be associated to the Schrödinger equation for a spinless non relativistic particle. The Schrödinger equation

$$(1) \quad i\hbar \frac{\partial \Psi}{\partial t} = \frac{1}{2m} \left( \hbar \frac{\partial}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right)^2 \Psi + eA_0 \Psi,$$

can be replaced by a set of two real equations for the amplitude  $R$  and the

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<sup>(1)</sup> E. MADELUNG: *Zeits. f. Phys.*, **40**, 332 (1926).



phase  $S$  of the wave function  $\Psi$

$$(2) \quad \Psi = R \exp \left[ \frac{i}{\hbar} S \right],$$

$$(3) \quad \begin{cases} \frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right)^2 + eA_0 - \frac{\hbar^2}{2m} \frac{\Delta R}{R} = 0, \\ \frac{\partial R^2}{\partial t} + \operatorname{div} \left( \frac{R^2}{m} \left( \frac{\partial S}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right) \right) = 0. \end{cases}$$

By introducing the velocity field

$$(4) \quad \mathbf{v} = \frac{1}{m} \left( \frac{\partial S}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right),$$

and taking the gradients of both sides of the first equation (3), we get the Euler equation of the Madelung fluid

$$(5) \quad m \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \mathbf{v} = e \left( \mathbf{E} + \frac{\mathbf{v}}{c} \wedge \mathbf{H} \right) + \frac{\hbar^2}{2m} \frac{\partial}{\partial \mathbf{x}} \frac{\Delta R}{R}.$$

$\mathbf{E}$  and  $\mathbf{H}$  denote the electric and magnetic fields, respectively. In the Euler equation (5) there is a non electromagnetic force corresponding to the potential  $U$

$$(6) \quad U = - \frac{\hbar^2}{2m} \frac{\Delta R}{R}.$$

$U$  is called usually the quantum potential. It plays a central part in the DE BROGLIE <sup>(2)</sup> pilot wave theory and in the deterministic interpretation of the quantum mechanics discussed recently by BOHM <sup>(3)</sup>.

It is well known that the Madelung model does not lead to a satisfactory interpretation of the Schrödinger equation. By considering the Madelung formalism as the classical theory of the motion of a fluid medium and applying to it the second quantization we can get a satisfactory interpretation. We shall discuss the second quantization of the Madelung theory in section 3.

*The main objective of the present paper is the derivation of an improved form of hydrodynamical model from the usual interpretation of the Schrödinger equation.*

<sup>(2)</sup> L. DE BROGLIE: *Compt. Rend.*, **183**, 447 (1926); **184**, 273 (1927); **185**, 380 (1927).

<sup>(3)</sup> D. BOHM: *Phys. Rev.*, **85**, 166 (1952); **85**, 180 (1952); **89**, 458 (1953).

The Madelung model describes the mean motion of a turbulent fluid, more exactly, of the stochastic model of a turbulent medium, as we shall see.

In order to obtain the stochastic model of a turbulent medium from the quantum mechanics, we shall introduce operators for the densities of charge, current, mass, momentum, etc., in the quantum mechanics of a point particle. In the classical theory of the motion of a point particle, we have delta-like charge and current densities  $e\rho(t, \mathbf{x})$  and  $e\mathbf{j}(t, \mathbf{x})$

$$(7) \quad \rho(t, \mathbf{x}) = \delta(\mathbf{x}(t) - \mathbf{x}), \quad \mathbf{j}(t, \mathbf{x}) = \delta(\mathbf{x}(t) - \mathbf{x})\dot{\mathbf{x}}(t).$$

The operators for the charge and current densities, in the Schrödinger representation of the quantum mechanics, can be obtained by the application of the standard rules

$$(8) \quad \rho(\mathbf{x})_{\text{op}} = \delta(\mathbf{x}_{\text{op}} - \mathbf{x}), \quad \mathbf{j}(t, \mathbf{x})_{\text{op}} = \frac{1}{2} \{ \delta(\mathbf{x}_{\text{op}} - \mathbf{x}), \dot{\mathbf{x}}(t)_{\text{op}} \},$$

$$(9) \quad \dot{\mathbf{x}}(t)_{\text{op}} = \frac{1}{m} \left( \mathbf{p}_{\text{op}} - \frac{e}{c} \mathbf{A}(t, \mathbf{x}_{\text{op}}) \right),$$

$$(10) \quad \{a_{\text{op}}, b_{\text{op}}\} = [a_{\text{op}}, b_{\text{op}}]_+ = a_{\text{op}}b_{\text{op}} + b_{\text{op}}a_{\text{op}}.$$

It is easily seen that the expectation values of the operators  $\rho(\mathbf{x})_{\text{op}}$  and  $\mathbf{j}(t, \mathbf{x})_{\text{op}}$  in the state of the particle defined by the wave function  $\Psi(t)$  coincide with the charge and current densities in the Madelung model

$$(11) \quad \langle \rho(\mathbf{x})_{\text{op}} \rangle_{\Psi} = R^2(t, \mathbf{x}) = \rho(t, \mathbf{x})_{\text{M}}, \quad \langle \mathbf{j}(t, \mathbf{x})_{\text{op}} \rangle_{\Psi} = R^2(t, \mathbf{x})\mathbf{v}(t, \mathbf{x}).$$

The mass density operator is  $m\rho(\mathbf{x})_{\text{op}}$  and the operator vector for the density of mechanical momentum is  $\mathbf{G}(t, \mathbf{x})_{\text{op}}$

$$(12) \quad \mathbf{G}(t, \mathbf{x})_{\text{op}} = m\mathbf{j}(t, \mathbf{x})_{\text{op}}, \quad \langle \mathbf{G}(t, \mathbf{x})_{\text{op}} \rangle_{\Psi} = \mathbf{G}(t, \mathbf{x})_{\text{M}}.$$

In the classical theory of the point particle, we can introduce the spatial components of the stress tensor  $T_{ab}(t, \mathbf{x})$

$$(13) \quad T_{ab}(t, \mathbf{x}) = m\delta(\mathbf{x}(t) - \mathbf{x})\dot{x}_a(t)\dot{x}_b(t).$$

The operators for the spatial components of the stress tensor in the Schrödinger representation will be taken as follows

$$(14) \quad T_{ab}(t, \mathbf{x})_{\text{op}} = \\ = \frac{m}{2} (\dot{x}_a(t)_{\text{op}} \delta(\mathbf{x}_{\text{op}} - \mathbf{x}) \dot{x}_b(t)_{\text{op}} + \dot{x}_b(t)_{\text{op}} \delta(\mathbf{x}_{\text{op}} - \mathbf{x}) \dot{x}_a(t)_{\text{op}}) - \frac{\hbar^2}{4m} \delta_{ab} \left( \frac{\partial}{\partial \mathbf{x}} \right)^2 \delta(\mathbf{x}_{\text{op}} - \mathbf{x}).$$

The justification of this choice will be given in section 2. It is important to notice the existence of the last term in the right hand side of (14), which has no classical analogue. This term may be regarded as a non kinetic part of the stress tensor operators, i.e. as being due to some kind of elastic tensions. It is easily seen that the expectation values of the  $T_{ab}(t, \mathbf{x})_{op}$  coincide with the ordinary expressions for the spatial components of the stress tensor associated to the Schrödinger equation

$$(15) \quad \langle T_{ab}(t, \mathbf{x})_{op} \rangle_\Psi = -\frac{1}{2m} \left( \hbar i \frac{\partial}{\partial x_a} + \frac{e}{c} A_a \right) \Psi^* \left( \hbar i \frac{\partial}{\partial x_b} - \frac{e}{c} A_b \right) \Psi - \frac{1}{2m} \left( \hbar i \frac{\partial}{\partial x_b} + \frac{e}{c} A_b \right) \Psi^* \left( \hbar i \frac{\partial}{\partial x_a} - \frac{e}{c} A_a \right) \Psi - \frac{\hbar^2}{4m} \delta_{ab} \Delta R^2 = T_{ab}(t, \mathbf{x})_M.$$

The Euler equations for the Madelung fluid follow directly from the equations

$$(16) \quad \frac{\partial}{\partial t} G_a(t, \mathbf{x})_M + \sum_{b=1}^3 \frac{\partial}{\partial x_b} T_{ab}(t, \mathbf{x})_M = e R^2 \left( E_a + \left( \frac{\mathbf{v}}{c} \wedge \mathbf{H} \right)_a \right),$$

since we get from (15)

$$(17) \quad T_{ab}(t, \mathbf{x})_M = m R^2 v_a v_b + \frac{\hbar^2}{m} \frac{\partial R}{\partial x_a} \frac{\partial R}{\partial x_b} - \frac{\hbar^2}{4m} \delta_{ab} \Delta R^2,$$

so that

$$(18) \quad \frac{\partial}{\partial t} G_a(t, \mathbf{x})_M + \sum_{b=1}^3 \frac{\partial}{\partial x_b} T_{ab}(t, \mathbf{x})_M = m R^2 \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) v_a + R^2 \frac{\partial U}{\partial x_a}.$$

In the ordinary stochastic theory of the turbulent motion of a fluid, the various dynamical quantities are represented by random functions satisfying the Navier-Stokes equations. The mean motion of the turbulent fluid corresponds to the expectation values of the random dynamical quantities. In quantum mechanics, the densities of charge, current, mass and momentum in a quantal state may be regarded as random functions which take only delta-like values. Thus the quantal motion of the point particle is associated to a kind of turbulent motion of a medium, whose mean motion coincides with that of the Madelung fluid.

The preceding results allow us to understand the meaning of the Bohm trajectories, i.e. of the classical trajectories associated to the first equation (3), viewed as a Hamilton-Jacobi equation:

*The Bohm trajectories are those of the mean motion of the turbulent medium associated to the quantal motion of the point particle.*

By considering the Heisenberg representation, we can get the analogue of the stochastic equations of the theory of the turbulent fluids. In the Heisenberg representation, the operators for the charge and current densities satisfy the continuity equation

$$(19) \quad \frac{\partial}{\partial t} \varrho(t, \mathbf{x})_{\text{op}}^H + \text{div } \mathbf{j}(t, \mathbf{x})_{\text{op}}^H = 0,$$

and the operators for the momentum density and stress tensor components satisfy the equations

$$(20) \quad \frac{\partial}{\partial t} G_a(t, \mathbf{x})_{\text{op}}^H + \sum_{b=1}^3 \frac{\partial}{\partial x_b} T_{ab}(t, \mathbf{x})_{\text{op}}^H = \frac{1}{2} \{f_a(t)_{\text{op}}^H, \varrho(t, \mathbf{x})_{\text{op}}^H\},$$

$$(21) \quad \mathbf{f}(t)_{\text{op}}^H = e\mathbf{E}(t, \mathbf{x}(t)_{\text{op}}^H) + \frac{e}{2c} (\dot{\mathbf{x}}(t)_{\text{op}}^H \wedge \mathbf{H}(t, \mathbf{x}(t)_{\text{op}}^H) + \text{hermitian conjugate}).$$

Equations (19) and (20) correspond to the stochastic equations in the theory of the turbulent fluids. Equation (16) can be obtained from (20), by taking the expectation values of both sides. Thereby (20) may be regarded as the operator form of the Euler equations.

It is well known that the elastic stress tensor of the mean motion of a turbulent fluid does not coincide with the expectation value of the stochastic elastic stress tensor, because of the contribution of the turbulence to the transfer of momentum which gives rise to the Reynolds terms (see J. E. MOYAL: *Stochastic processes and statistical physics*, in *Journ. of the Roy. Stat. Soc.*, series B, **11**, 193 (1949)). The quantum potential, that appears in the equations of the mean motion, results from the combination of contributions of the kinetic part of  $T_{ab}(t, \mathbf{x})_{\text{op}}$  with its elastic part  $-(\hbar^2/4m) \delta_{ab} (\partial/\partial \mathbf{x})^2 \varrho(\mathbf{x})_{\text{op}}$ . The Reynolds term in  $T_{ab}(t, \mathbf{x})_M$  is the second term in the right hand side of (17). Hence:

*The quantum potential  $U$  results from the combination of the turbulent transfer of momentum, described by the term  $(\hbar^2/m)(\partial R/\partial x_a)(\partial R/\partial x_b)$  in  $T_{ab}(t, \mathbf{x})_M$ , with the effect of the elastic tension described by  $-(\hbar^2/4m) \delta_{ab} \Delta R^2$ .*

The density operator formalism can be immediately generalized to a particle with spin. We shall denote the spin operator vector by  $\mathbf{\Sigma}_{\text{op}}$  and the intrinsic magnetic moment by  $\lambda \mathbf{\Sigma}_{\text{op}}$ . The operators for the densities of charge and current, the density of momentum and the spatial components of the stress tensor are the same as above. Because of the spin term in the hamiltonian

$$(22) \quad H_{\text{op}} = \frac{1}{2m} \left( \mathbf{p}_{\text{op}} - \frac{e}{c} \mathbf{A} \right)^2 + eA_0 - \lambda (\mathbf{H} \cdot \mathbf{\Sigma}_{\text{op}}),$$



the operator form of the Euler equation becomes

$$(23) \quad \frac{\partial}{\partial t} G_a(t, \mathbf{x})_{\text{op}}^H + \sum_{b=1}^3 \frac{\partial}{\partial x_b} T_{ab}(t, \mathbf{x})_{\text{op}}^H = \frac{1}{2} \{f'_a(t)_{\text{op}}^H, \varrho(t, \mathbf{x})_{\text{op}}^H\},$$

$$(24) \quad f'(t)_{\text{op}}^H = f(t)_{\text{op}}^H + \lambda \left( \frac{\partial}{\partial \xi} (\mathbf{H}(t, \xi) \cdot \Sigma(t)_{\text{op}}^H) \right)_{(\xi = \mathbf{x}(t)_{\text{op}}^H)}.$$

The operator vector for the density of spin in the Schrödinger representation will be denoted by  $\sigma(\mathbf{x})_{\text{op}}$

$$(25) \quad \sigma(\mathbf{x})_{\text{op}} = \Sigma_{\text{op}} \varrho(\mathbf{x})_{\text{op}}.$$

The spin density operator vector in the Heisenberg representation satisfies the following conservation equation

$$(26) \quad \frac{\partial}{\partial t} \sigma(t, \mathbf{x})_{\text{op}}^H = \lambda \sigma(t, \mathbf{x})_{\text{op}}^H \wedge \mathbf{H}(t, \mathbf{x}(t)_{\text{op}}^H) - \Sigma(t)_{\text{op}}^H \operatorname{div} \mathbf{j}(t, \mathbf{x})_{\text{op}}^H.$$

In the present case, the Madelung fluid possesses, besides the ordinary inertia, a gyrostatic inertia, arising from the spin. At each point of the fluid the spin vector  $\Sigma(t, \mathbf{x})$ , is defined as follows

$$(27) \quad \Sigma(t, \mathbf{x}) = \langle \sigma(\mathbf{x})_{\text{op}} \rangle_{\Psi} / \langle \varrho(\mathbf{x})_{\text{op}} \rangle_{\Psi}.$$

It follows from (26), by taking the expectation values of both sides, that

$$(28) \quad \varrho_M \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \Sigma_a = \lambda \varrho_M (\Sigma \wedge \mathbf{H})_a + \operatorname{div} (\varrho_M \mathbf{v} \Sigma_a) - \operatorname{div} \langle \mathbf{j}_{\text{op}} \Sigma_{a, \text{op}} \rangle_{\Psi},$$

$$(29) \quad \mathbf{v}(t, \mathbf{x}) = \langle \mathbf{j}(t, \mathbf{x})_{\text{op}} \rangle_{\Psi} / \langle \varrho(\mathbf{x})_{\text{op}} \rangle_{\Psi}.$$

The case of the spin 1/2 is particularly interesting, because the vector  $\Sigma(t, \mathbf{x})$  has a constant magnitude  $\hbar/2$ . For higher values of the spin,  $|\Sigma(t, \mathbf{x})|$  varies, in general, with the position. *The last two terms in the right hand side of (28) correspond to a quantal torque analogous to the force derived from the quantum potential.*

The behaviour of the Madelung fluid in the case of a particle with spin is more complex than in that of a spinless particle. It will be shown in section 2 that, for a particle of spin  $s$ , the Madelung fluid may be treated as a mixture of  $2s+1$  phases which exchange mass under the action of a magnetic field. The velocity  $\mathbf{v}(t, \mathbf{x})$  may be defined, as in the case of a spinless particle, as the ratio of the expectation values of the current and charge densities. It may

be viewed as a center of mass velocity of the  $2s+1$  phases at each point. In the absence of magnetic fields, the motion of the individual phases is irrotational, but, in general,  $\text{rot } \mathbf{v}$  will not vanish. The Euler equation can be obtained from (23) by taking the expectation values of both sides

$$(30) \quad m \varrho_M \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \mathbf{v} = e \varrho_M \left( \mathbf{E} + \frac{\mathbf{v}}{c} \wedge \mathbf{H} \right) + \lambda \sum_{a=1}^3 \frac{\partial H_a}{\partial \mathbf{x}} \cdot \Sigma_a + \\ + \frac{\hbar^2}{2m} \sum_r R_r^2 \frac{\partial}{\partial \mathbf{x}} \frac{\Delta R_r}{R_r} - m \sum_{a=1}^3 \frac{\partial}{\partial x_a} \sum_r R_r^2 (v_{r,a} - v_a)(\mathbf{v}_r - \mathbf{v}),$$

$$(31) \quad R_r = |\Psi_r|, \quad \frac{S_r}{\hbar} = \text{phase of } \Psi_r,$$

$$(32) \quad \mathbf{v}_r = \frac{1}{m} \left( \frac{\partial S_r}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right).$$

The quantum potential of the case of a spinless particle is now replaced by the third group of terms in the right hand side of (30), and there is the fourth group of terms which arises from the relative motions of the different phases with respect to the center of mass, in each element of volume.

Equation (17) is now replaced by the following

$$(33) \quad \langle T_{ab}(t, \mathbf{x})_{\text{op}} \rangle_{\Psi} = m \varrho_M v_a v_b + m \sum_r R_r^2 (v_{r,a} - v_a)(v_{r,b} - v_b) + \\ + \frac{\hbar^2}{m} \sum_r \frac{\partial R_r}{\partial x_a} \frac{\partial R_r}{\partial x_b} - \frac{\hbar^2}{4m} \delta_{ab} \Delta (\sum_r R_r^2).$$

The Reynolds terms arising from the turbulence are  $m \sum_r R_r^2 (v_{r,a} - v_a)(v_{r,b} - v_b) + (\hbar^2/m) \sum_r (\partial R_r / \partial x_a)(\partial R_r / \partial x_b)$ .

In the case of a particle with spin, the Euler equation (30) and the equation of continuity do not constitute a system equivalent to the Schrödinger equation. In order to get a system equivalent to the Schrödinger equation, we may take the Euler equations and the continuity equations for the  $2s+1$  phases

$$(34) \quad m \left( \frac{\partial}{\partial t} + \mathbf{v}_r \cdot \frac{\partial}{\partial \mathbf{x}} \right) \mathbf{v}_r = e \left( \mathbf{E} + \frac{\mathbf{v}_r}{c} \wedge \mathbf{H} \right) + \frac{\hbar^2}{2m} \frac{\partial}{\partial \mathbf{x}} \frac{\Delta R_r}{R_r} + \\ + \frac{\partial}{\partial \mathbf{x}} \left( \left\{ \frac{\lambda}{2R_r^2} (\Psi^* \Sigma_{\text{op}})_r \Psi_r + \Psi_r^* (\Sigma_{\text{op}})_r \Psi_r \right\} \cdot \mathbf{H} \right),$$

$$(35) \quad \frac{\partial R_r^2}{\partial t} + \text{div} (R_r^2 \mathbf{v}_r) = \frac{i\lambda}{\hbar} \mathbf{H} \cdot (\Psi_r^* (\Sigma_{\text{op}})_r \Psi_r - (\Psi^* \Sigma_{\text{op}})_r \Psi_r).$$

BOHM, SCHILLER and TIOMNO (4) have discussed the hydrodynamical equations for the case of spin 1/2 and magnetic moment  $e\hbar/2mc$ . In this case, the Pauli equation can be replaced by the Euler equation for the Madelung fluid, together with the continuity equation and the equations of motion of the spin vector  $\Sigma(t, \mathbf{x})$ . The case of spin 1/2 has special properties, because the wave spinor can be expressed in terms of the density  $R^2$  and the three Euler angles  $\theta, \varphi, \chi$

$$(36) \quad \Psi_1 = R \cos \frac{\theta}{2} \exp \left[ \frac{i}{2} (\varphi + \chi) \right], \quad \Psi_2 = iR \sin \frac{\theta}{2} \exp \left[ \frac{i}{2} (\varphi - \chi) \right].$$

The components of  $\Sigma$  can be expressed in terms of the Euler angles  $\theta, \chi$

$$(37) \quad \Sigma_1 = \frac{\hbar}{2} \sin \theta \sin \chi, \quad \Sigma_2 = \frac{\hbar}{2} \sin \theta \cos \chi, \quad \Sigma_3 = \frac{\hbar}{2} \cos \theta,$$

and we have:

$$(38) \quad m\mathbf{v} = \frac{\hbar}{2} \left( \frac{\partial \varphi}{\partial \mathbf{x}} + \cos \theta \frac{\partial \chi}{\partial \mathbf{x}} \right) - \frac{e}{c} \mathbf{A}.$$

The Pauli equation gives a set of equations for the four variables  $R, \theta, \varphi, \chi$ . The equation for  $R$  is that of continuity. The equations for  $\theta$  and  $\chi$  are equivalent to those for the spin vector  $\Sigma(t, \mathbf{x})$  and, associated to the equation for  $\varphi$ , they lead to the Euler equation for  $\mathbf{v}$ .

Equation (4) shows that the motions of the Madelung fluid associated with the Schrödinger equation are of a particular kind, because the vorticity  $\zeta$  satisfies the condition

$$(39) \quad \zeta = - \frac{e\mathbf{H}}{mc}, \quad (\zeta = \text{rot } \mathbf{v}).$$

In the case of an inviscid barotropic charged fluid, the magnetic field creates vorticity. It is easy to extend the Helmholtz vortex theory to the vortex motions of the barotropic charged fluids whose charge and mass densities have a constant ratio  $k$ . It suffices to replace the vorticity  $\zeta$  by the vec-

(4) D. BOHM, R. SCHILLER and J. TIOMNO: Unpublished work.

tor  $\boldsymbol{\eta}$

$$(40) \quad \boldsymbol{\eta} = \boldsymbol{\zeta} + \frac{k}{c} \mathbf{H}.$$

The Madelung fluid is not a barotropic perfect fluid, but it is possible to develop a theory of the vortex motions of the Madelung fluid, by modifying slightly the usual hydrodynamic proofs. Thus there is the analogue of the Kelvin theorem of circulation

$$(41) \quad \int_C \mathbf{u} \cdot \delta \mathbf{x} = \int_{C_0} \mathbf{u}_0 \cdot \delta \mathbf{x}_0, \quad \mathbf{u} = \mathbf{v} + \frac{e}{mc} \mathbf{A},$$

$C$  being the closed curve formed at the time  $t$  by the elements of the fluid that where on the closed curve  $C_0$  at the initial time  $t_0$ ;  $\mathbf{x}_0$  being the the position at the time  $t_0$  of the element of fluid and  $\mathbf{u}_0$  the corresponding  $\mathbf{u}$ -vector. The form of the circulation theorem shows that the vorticity is now replaced by  $\boldsymbol{\eta} = \text{rot } \mathbf{u}$ . Therefore, the motions in which  $\boldsymbol{\eta} = 0$  correspond to the irrotational motions of the inviscid fluids. We shall call them quasi-irrotational. *The motions of the Madelung fluid associated with the Schrödinger equation being quasi-irrotational, the more general motions must correspond to a generalization of the Schrödinger equation.* This generalization of the Schrödinger equation was already discussed, by us from a different point of view, in a recent paper <sup>(5)</sup>.

Equation (32) shows that the motions of the  $2s+1$  phases of the fluid associated to a particle with spin  $s$  are also quasi-irrotational. It would also be possible to discuss more general motions of those phases. A generalization of the Schrödinger equation for a particle with spin can also be obtained in this way.

The quantization of the quasi-irrotational motions of the Madelung fluid is equivalent to the second quantization for non interacting particles. It is, of course, possible to obtain the interactions between the particles, by considering a quantized electro-magnetic field in interaction with the quantized fluid. The second quantization of the Schrödinger equation introduces a quantized field whose quanta are the particles in consideration. By means of the quantized wave functions  $\psi_r(\mathbf{x})_{\text{op}}$  and their conjugate operators, that satisfy the well known commutation rules

$$(42) \quad [\psi_r(\mathbf{x})_{\text{op}}, \psi_{r'}(\mathbf{x}')_{\text{op}}]_{\pm} = 0, \quad [\psi_r(\mathbf{x})_{\text{op}}, \psi_{r'}^*(\mathbf{x}')_{\text{op}}]_{\pm} = \delta_{rr'} \delta(\mathbf{x} - \mathbf{x}'),$$

we can form the operators for the density of charge  $\varrho^{(2)}(\mathbf{x})_{\text{op}}$ , for the density

(5) M. SCHÖNBERG: *Nuovo Cimento*, **11**, 674 (1954).



of current  $\mathbf{j}^{(2)}(t, \mathbf{x})_{\text{op}}$ , for the spatial components of the stress-tensor  $T_{ab}^{(2)}(t, \mathbf{x})_{\text{op}}$

$$(43) \quad \left\{ \begin{aligned} \varrho^{(2)}(\mathbf{x})_{\text{op}} &= \psi^*(\mathbf{x})_{\text{op}} \psi(\mathbf{x})_{\text{op}}, \\ \mathbf{j}^{(2)}(t, \mathbf{x})_{\text{op}} &= \frac{1}{2m} \left( \psi^*(\mathbf{x})_{\text{op}} \left( \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right) \psi(\mathbf{x})_{\text{op}} - \left( \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} + \frac{e}{c} \mathbf{A} \right) \psi^*(\mathbf{x})_{\text{op}} \cdot \psi(\mathbf{x})_{\text{op}} \right), \\ T_{ab}^{(2)}(t, \mathbf{x})_{\text{op}} &= -\frac{1}{2m} \left\{ \left( \frac{\hbar}{i} \frac{\partial}{\partial x_a} + \frac{e}{c} A_a \right) \psi^*(\mathbf{x})_{\text{op}} \left( \frac{\hbar}{i} \frac{\partial}{\partial x_b} - \frac{e}{c} A_b \right) \psi(\mathbf{x})_{\text{op}} + \right. \\ &\quad \left. + \left( \frac{\hbar}{i} \frac{\partial}{\partial x_b} + \frac{e}{c} A_b \right) \psi^*(\mathbf{x})_{\text{op}} \left( \frac{\hbar}{i} \frac{\partial}{\partial x_a} - \frac{e}{c} A_a \right) \psi(\mathbf{x})_{\text{op}} \right\} - \frac{\hbar^2}{4m} \delta_{ab} \Delta \varrho^{(2)}(\mathbf{x})_{\text{op}}. \end{aligned} \right.$$

The sign  $+$  and  $-$  in the commutation rules (42) correspond to the Fermi and Bose statistics, respectively. In the ordinary treatment of the second quantization, the above operators (43) are not derived from operators of the one particle theory. Our formalism for the one particle densities allows us to obtain the operators (43) by the application of the standard rule

$$(44) \quad \left\{ \begin{aligned} \varrho^{(2)}(\mathbf{x})_{\text{op}} &= \int \psi^*(\mathbf{x}')_{\text{op}} \varrho(\mathbf{x})_{\text{op}} \psi_{\text{op}}(\mathbf{x}') d\mathbf{x}', \\ \mathbf{j}^{(2)}(t, \mathbf{x})_{\text{op}} &= \int \psi^*(\mathbf{x}')_{\text{op}} \mathbf{j}(t, \mathbf{x})_{\text{op}} \psi_{\text{op}}(\mathbf{x}') d\mathbf{x}', \\ T_{ab}^{(2)}(t, \mathbf{x})_{\text{op}} &= \int \psi^*(\mathbf{x}')_{\text{op}} T_{ab}(t, \mathbf{x})_{\text{op}} \psi_{\text{op}}(\mathbf{x}') d\mathbf{x}'. \end{aligned} \right.$$

The one quantum states of the quantized Madelung fluid are essentially the same as the particle states of the one particle formalism. *The Madelung fluid corresponds to the expectation values of the quantities of the quantized fluid in the one quantum states. The quantization of the motion of the fluid regenerates the turbulence eliminated by taking the expectation values in the one particle formalism.* This point is discussed in section 3.

The regeneration of the turbulence can be related to the vacuum fluctuations of the quantized Madelung fluid, as will be shown in section 4. We can get the values of the quantities in a given state of motion of the Madelung fluid, by taking the average values in different states of motion of the quantized fluid. Although the turbulence be different in those states, the mean motions coincide. In the case of the Bose statistics, the absorption operators commute at all the points of space and there are states in which they all have defined values. These states of the quantized fluid are closely related to the motions of the unquantized fluid, even more than the one quantum states, because, not only quantities of the unquantized fluid such as the charge and current densities and the components of the stress tensor, but even  $\Psi$  can

be obtained as expectation values of the operators of the quantized fluid in those states.

The quantized Madelung fluid gives the best form of the turbulent hydrodynamical model. It may seem that the quantized fluid model be uninteresting, since a quantized fluid appears as a more complex quantal structure than a quantal particle. It is, however, necessary to notice that the so-called «second quantization» is not essentially a quantal method, as we have already shown <sup>(6)</sup>. Even the classical particles may be regarded as quanta of a field in a six dimensional space <sup>(7)</sup>. The limitations introduced by the uncertainty principle allow precisely the introduction of a three dimensional model, instead of the abstract six dimensional (phase-space) «quantized» field description of the classical particles. The fact that the quantization of the fluid is not dependent on the Planck constant is clearly shown by the commutation rules (42), that do not involve  $\hbar$ .

## 2. — The density operators.

We shall now derive the equations (11) of section 1. We shall treat the more general case of a particle with spin  $s$ , in which there is a one column wave function matrix with  $2s+1$  rows. The elements of the wave function matrix will be denoted by  $\Psi_r(t, \mathbf{x})$ . We have

$$(1) \quad \langle \varrho(\mathbf{x})_{\text{op}} \rangle_{\Psi} = \sum_r \int \Psi_r^*(t, \mathbf{x}') \delta(\mathbf{x}_{\text{op}} - \mathbf{x}) \Psi_r(t, \mathbf{x}') d\mathbf{x}' = \\ = \sum_r \int \Psi_r^*(t, \mathbf{x}') \delta(\mathbf{x}' - \mathbf{x}) \Psi_r(t, \mathbf{x}') d\mathbf{x}' = \sum_r |\Psi_r(t, \mathbf{x})|^2,$$

$$(2) \quad \langle \mathbf{j}(t, \mathbf{x})_{\text{op}} \rangle_{\Psi} = \frac{1}{2} \sum_r \int \Psi_r^*(t, \mathbf{x}') \{ \dot{\mathbf{x}}(t, \mathbf{x}_{\text{op}} - \mathbf{x}), \dot{\mathbf{x}}(t, \mathbf{x}_{\text{op}}) \} \Psi_r(t, \mathbf{x}') d\mathbf{x}' = \\ = \frac{1}{2m} \sum_r \Psi_r^*(t, \mathbf{x}) \left( \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A}(t, \mathbf{x}) \right) \Psi_r(t, \mathbf{x}) + \\ + \frac{1}{2m} \sum_r \int \Psi_r^*(t, \mathbf{x}') \left( \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A}(t, \mathbf{x}') \right) (\delta(\mathbf{x}' - \mathbf{x}) \Psi_r(t, \mathbf{x}')) d\mathbf{x}' = \\ = \frac{1}{2m} \sum_r \Psi_r^*(t, \mathbf{x}) \left( \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A}(t, \mathbf{x}) \right) \Psi_r(t, \mathbf{x}) - \\ - \frac{1}{2m} \sum_r \Psi_r(t, \mathbf{x}) \left( \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} + \frac{e}{c} \mathbf{A}(t, \mathbf{x}) \right) \Psi_r^*(t, \mathbf{x}).$$

<sup>(6)</sup> M. SCHÖNBERG: *Nuovo Cimento*, **10**, 697 (1953).

<sup>(7)</sup> M. SCHÖNBERG: *Nuovo Cimento*, **9**, 1139 (1952); **10**, 419 (1953).

We shall denote by  $R_r$  the amplitude and by  $S_r/\hbar$  the phase of  $\Psi_r$

$$(3) \quad \Psi_r = R_r \exp \left[ \frac{i}{\hbar} S_r \right].$$

We shall also introduce the velocity fields  $\mathbf{v}_r$ ,

$$(4) \quad \mathbf{v}_r(t, \mathbf{x}) = \frac{1}{m} \left( \frac{\partial}{\partial \mathbf{x}} S_r(t, \mathbf{x}) - \frac{e}{c} \mathbf{A}(t, \mathbf{x}) \right).$$

It is easily seen that

$$(5) \quad \begin{cases} \langle \varrho(\mathbf{x})_{\text{op}} \rangle_{\Psi} = \sum_r R_r^2(t, \mathbf{x}) = \varrho(t, \mathbf{x})_M, \\ \langle \mathbf{j}(t, \mathbf{x})_{\text{op}} \rangle_{\Psi} = \sum_r R_r^2(t, \mathbf{x}) \mathbf{v}_r(t, \mathbf{x}) = \varrho(t, \mathbf{x})_M \mathbf{v}(t, \mathbf{x}). \end{cases}$$

Let us consider an operator  $O_{\text{op}}$  which is a function of the time  $t$ , the coordinate and momentum operators  $x_{a,\text{op}}$  and  $p_{a,\text{op}}$  and the spin components  $\Sigma_{a,\text{op}}$ . The time derivative of the corresponding operator in the Heisenberg representation  $O(t)_{\text{op}}^H$  is given by the well known equation

$$(6) \quad \frac{\partial}{\partial t} O(t)_{\text{op}}^H = \left( \frac{\partial}{\partial t} O(t)_{\text{op}}^H \right)_d - \frac{i}{\hbar} [O(t)_{\text{op}}^H, H(t)_{\text{op}}^H],$$

$((\partial/\partial t)O(t)_{\text{op}}^H)_d$  denotes the derivative computed by taking into account only the direct dependence on the time. It follows from (6) that

$$(7) \quad \begin{aligned} \frac{\partial}{\partial t} \varrho(t, \mathbf{x})_{\text{op}}^H &= -\frac{i}{\hbar} \left[ \varrho(t, \mathbf{x})_{\text{op}}^H, \frac{1}{2m} \left( \mathbf{p}(t)_{\text{op}}^H - \frac{e}{c} \mathbf{A}(t, \mathbf{x}(t)_{\text{op}}^H) \right)^2 \right] = \\ &= -\frac{i}{2\hbar m} \left\{ \left( \mathbf{p}(t)_{\text{op}}^H - \frac{e}{c} \mathbf{A}(t, \mathbf{x}(t)_{\text{op}}^H) \right), [\partial(\mathbf{x} - \mathbf{x}(t)_{\text{op}}^H), \mathbf{p}(t)_{\text{op}}^H] \right\} = \\ &= -\frac{i}{2\hbar} \{ \dot{\mathbf{x}}(t)_{\text{op}}^H, [\partial(\mathbf{x} - \mathbf{x}(t)_{\text{op}}^H), \mathbf{p}(t)_{\text{op}}^H] \}. \end{aligned}$$

By taking into account the well known formula

$$(8) \quad [F(\mathbf{x}_{\text{op}}), p_{a,\text{op}}] = i\hbar \frac{\partial F_{\text{op}}}{\partial x_{a,\text{op}}},$$

we get

$$(9) \quad \frac{\partial}{\partial t} \varrho(t, \mathbf{x})_{\text{op}}^H = -\frac{1}{2} \dot{\mathbf{x}}(t)_{\text{op}}^H \cdot \frac{\partial}{\partial \mathbf{x}} \delta(\mathbf{x} - \mathbf{x}(t)_{\text{op}}^H) - \frac{1}{2} \frac{\partial}{\partial \mathbf{x}} \delta(\mathbf{x} - \mathbf{x}(t)_{\text{op}}^H) \cdot \dot{\mathbf{x}}(t)_{\text{op}}^H = \\ = -\frac{1}{2} \operatorname{div} \{ \dot{\mathbf{x}}(t)_{\text{op}}^H, \delta(\mathbf{x} - \mathbf{x}(t)_{\text{op}}^H) \} = -\operatorname{div} \mathbf{j}(t, \mathbf{x})_{\text{op}}^H.$$

Thus we have obtained the equation (19) of section 1.

We shall now derive equation (23) of section 1. We have

$$(10) \quad \frac{\partial}{\partial t} \mathbf{G}(t, \mathbf{x})_{\text{op}}^H = \frac{1}{2} \left\{ \frac{\partial}{\partial t} \left( \mathbf{p}(t)_{\text{op}}^H - \frac{e}{c} \mathbf{A}(t, \mathbf{x}(t)_{\text{op}}^H) \right), \delta(\mathbf{x} - \mathbf{x}(t)_{\text{op}}^H) \right\} + \\ + \frac{1}{2} \left\{ \left( \mathbf{p}(t)_{\text{op}}^H - \frac{e}{c} \mathbf{A}(t, \mathbf{x}(t)_{\text{op}}^H) \right), \frac{\partial}{\partial t} \delta(\mathbf{x} - \mathbf{x}(t)_{\text{op}}^H) \right\}.$$

It is easily seen that

$$(11) \quad \frac{\partial}{\partial t} \left( \mathbf{p}(t)_{\text{op}}^H - \frac{e}{c} \mathbf{A}(t, \mathbf{x}(t)_{\text{op}}^H) \right) = \mathbf{f}'(t)_{\text{op}}^H,$$

$\mathbf{f}'(t)_{\text{op}}^H$  being defined by the equations (24) and (21) of section 1. It follows from equations (10) and (11) that

$$(12) \quad \frac{\partial}{\partial t} \mathbf{G}(t, \mathbf{x})_{\text{op}}^H = \frac{1}{2} \{ \mathbf{f}'(t)_{\text{op}}^H, \varrho(t, \mathbf{x})_{\text{op}}^H \} + \frac{m}{2} \left\{ \dot{\mathbf{x}}(t)_{\text{op}}^H, \frac{\partial}{\partial t} \varrho(t, \mathbf{x})_{\text{op}}^H \right\}.$$

We get from the equation (14) of section 1

$$(13) \quad \sum_{b=1}^3 \frac{\partial}{\partial x_b} T_{ab}(t, \mathbf{x})_{\text{op}}^H = \frac{m}{2} \dot{x}_a(t)_{\text{op}}^H \frac{\partial}{\partial \mathbf{x}} \delta(\mathbf{x} - \mathbf{x}(t)_{\text{op}}^H) \cdot \dot{\mathbf{x}}(t)_{\text{op}}^H + \\ + \frac{m}{2} \dot{\mathbf{x}}(t)_{\text{op}}^H \cdot \frac{\partial}{\partial \mathbf{x}} \delta(\mathbf{x} - \mathbf{x}(t)_{\text{op}}^H) \dot{x}_a(t)_{\text{op}}^H - \frac{\hbar^2}{4m} \frac{\partial}{\partial x_a} \left( \frac{\partial}{\partial \mathbf{x}} \right)^2 \delta(\mathbf{x} - \mathbf{x}(t)_{\text{op}}^H).$$

It follows from (12), (13) and (9) that

$$(14) \quad \frac{\partial}{\partial t} G_a(t, \mathbf{x})_{\text{op}}^H + \sum_{b=1}^3 \frac{\partial}{\partial x_b} T_{ab}(t, \mathbf{x})_{\text{op}}^H = \frac{1}{2} \{ \mathbf{f}'(t)_{\text{op}}^H, \varrho(t, \mathbf{x})_{\text{op}}^H \} + \\ + \frac{m}{4} \left[ \dot{x}(t)_{\text{op}}^H, \sum_{b=1}^3 \frac{\partial}{\partial x_b} [ \varrho(t, \mathbf{x})_{\text{op}}^H, \dot{x}_b(t)_{\text{op}}^H ] \right] - \frac{\hbar^2}{4m} \frac{\partial}{\partial x_a} \left( \frac{\partial}{\partial \mathbf{x}} \right)^2 \varrho(t, \mathbf{x})_{\text{op}}^H,$$

but since

$$(15) \quad \sum_{b=1}^3 \left[ \frac{\partial}{\partial x_b} \varrho(t, \mathbf{x})_{\text{op}}^H, \dot{x}_b(t)_{\text{op}}^H \right] = \frac{1}{m} \sum_{b=1}^3 \frac{\partial}{\partial x_b} [\varrho(t, \mathbf{x})_{\text{op}}^H, p_b(t)_{\text{op}}^H] = -\frac{i\hbar}{m} \left( \frac{\partial}{\partial \mathbf{x}} \right)^2 \varrho(t, \mathbf{x})_{\text{op}}^H,$$

and

$$(16) \quad \begin{aligned} \frac{m}{4} \left[ \dot{x}_a(t)_{\text{op}}^H, \sum_{b=1}^3 \frac{\partial}{\partial x_b} [\varrho(t, \mathbf{x})_{\text{op}}^H, \dot{x}_b(t)_{\text{op}}^H] \right] = \\ = -\frac{i\hbar}{4} \left( \frac{\partial}{\partial \mathbf{x}} \right)^2 [\dot{x}_a(t)_{\text{op}}^H, \varrho(t, \mathbf{x})_{\text{op}}^H] = -\frac{i\hbar}{4m} \left( \frac{\partial}{\partial \mathbf{x}} \right)^2 [p_a(t)_{\text{op}}^H, \varrho(t, \mathbf{x})_{\text{op}}^H] = \\ = \frac{\hbar^2}{4m} \frac{\partial}{\partial x_a} \left( \frac{\partial}{\partial \mathbf{x}} \right)^2 \varrho(t, \mathbf{x})_{\text{op}}^H, \end{aligned}$$

the last two terms in the right hand side of (14) cancel. Hence

$$(17) \quad \frac{\partial}{\partial t} G_a(t, \mathbf{x})_{\text{op}}^H + \sum_{b=1}^3 \frac{\partial}{\partial x_b} T_{ab}(t, \mathbf{x})_{\text{op}}^H = \frac{1}{2} \{f'(t)_{\text{op}}^H, \varrho(t, \mathbf{x})_{\text{op}}^H\},$$

and thus we have obtained the equation (23) of section 1.

The operators  $T_{ab}(t, \mathbf{x})_{\text{op}}^H$  are symmetrical with respect to the indices  $a$  and  $b$ , hermitian and satisfy the equations (17), which have the right form. The non kinetic part of  $T_{ab}(t, \mathbf{x})_{\text{op}}^H$  disappears in the classical limit, because of the factor  $\hbar^2$ . It is possible to obtain non symmetrical hermitian operators  $T'_{ab}(t, \mathbf{x})_{\text{op}}^H$

$$(18) \quad T'_{ab}(t, \mathbf{x})_{\text{op}}^H = \frac{m}{4} \{ \dot{x}_a(t)_{\text{op}}^H, \{ \varrho(t, \mathbf{x})_{\text{op}}^H, \dot{x}_b \} \},$$

which satisfy an equation of the form (17) and can be derived from the classical stress tensor by ordering suitably the non commutable factors.

The second equation (3) of section 1 can be derived from the operator continuity equation, (19) of section 1, by taking expectation values. The wave functions  $\Psi_r(\mathbf{x})^H$  in the Heisenberg representation are time independent and satisfy the following equation, for any operator  $O_{\text{op}}$

$$(19) \quad \int \Psi^*(\mathbf{x})^H O(t)_{\text{op}}^H \Psi(\mathbf{x})^H d\mathbf{x} = \int \Psi^*(t, \mathbf{x}) O(t)_{\text{op}} \Psi(t, \mathbf{x}) d\mathbf{x},$$

$$(20) \quad \langle O(t)_{\text{op}}^H \rangle_{\Psi^H} = \langle O(t)_{\text{op}} \rangle_{\Psi}.$$



Hence we have

$$(21) \quad \left\langle \frac{\partial}{\partial t} O(t)_{\text{op}}^H \right\rangle_{\Psi^H} = \frac{\partial}{\partial t} \langle O(t)_{\text{op}} \rangle_{\Psi}.$$

By taking the expectation values of both sides of equation (19) of section 1 and applying (21) we get

$$(22) \quad \frac{\partial}{\partial t} \langle \varrho(\mathbf{x})_{\text{op}} \rangle_{\Psi} + \text{div} \langle \mathbf{j}(t, \mathbf{x})_{\text{op}} \rangle_{\Psi} = 0,$$

so that

$$(23) \quad \frac{\partial}{\partial t} (\sum_r R_r^2) + \text{div} (\sum_r R_r^2 \mathbf{v}_r) = \frac{\partial}{\partial t} \varrho(t, \mathbf{x})_M + \text{div} (\varrho(t, \mathbf{x})_M \mathbf{v}) = 0,$$

as a consequence of equations (5). In the particular case of a spinless particle we get the second equation (3) of section 1.

By taking the expectation values of both sides of (17) we get

$$(24) \quad \frac{\partial}{\partial t} \langle G_a(t, \mathbf{x})_{\text{op}} \rangle_{\Psi} + \sum_{b=1}^3 \frac{\partial}{\partial x_b} \langle T_{ab}(t, \mathbf{x})_{\text{op}} \rangle_{\Psi} = \frac{1}{2} \langle \{j'_a(t)_{\text{op}}, \varrho(\mathbf{x})_{\text{op}}\} \rangle_{\Psi}.$$

It is easily seen that

$$(25) \quad \langle G_a(t, \mathbf{x})_{\text{op}} \rangle_{\Psi} = m \sum_r R_r^2 \mathbf{v}_r = m \varrho(t, \mathbf{x})_M \mathbf{v}(t, \mathbf{x})$$

and

$$(26) \quad \begin{aligned} \langle T_{ab}(t, \mathbf{x})_{\text{op}} \rangle_{\Psi} &= -\frac{1}{2m} \sum_r \left( \frac{\hbar}{i} \frac{\partial}{\partial x_a} + \frac{e}{c} A_a \right) \Psi_r^* \left( \frac{\hbar}{i} \frac{\partial}{\partial x_b} - \frac{e}{c} A_b \right) \Psi_r - \\ &- \frac{1}{2m} \sum_r \left( \frac{\hbar}{i} \frac{\partial}{\partial x_b} + \frac{e}{c} A_b \right) \Psi_r^* \left( \frac{\hbar}{i} \frac{\partial}{\partial x_a} - \frac{e}{c} A_a \right) \Psi_r - \frac{\hbar^2}{4m} \delta_{ab} \Delta (\sum_r R_r^2) = \\ &= \frac{1}{m} \sum_r R_r^2 \left( \frac{\partial S_r}{\partial x_a} - \frac{e}{c} A_a \right) \left( \frac{\partial S_r}{\partial x_b} - \frac{e}{c} A_b \right) + \frac{\hbar^2}{m} \sum_r \frac{\partial R_r}{\partial x_a} \frac{\partial R_r}{\partial x_b} - \frac{\hbar^2}{4m} \delta_{ab} \Delta (\sum_r R_r^2) = \\ &= m \sum_r R_r^2 v_{r,a} v_{r,b} + \frac{\hbar^2}{m} \sum_r \frac{\partial R_r}{\partial x_a} \frac{\partial R_r}{\partial x_b} - \frac{\hbar^2}{4m} \delta_{ab} \Delta (\sum_r R_r^2). \end{aligned}$$

The expectation value in the right hand side of (24) is easily calculated

$$(27) \quad \begin{aligned} \langle \{f'(t)_{\text{op}}, \varrho(\mathbf{x})_{\text{op}}\} \rangle_{\Psi} &= \Psi^* f'(t)_{\text{op}} \Psi + (f'(t)_{\text{op}} \Psi)^* \Psi = \\ &= 2e \sum_r R_r^2 \left( \mathbf{E} + \frac{\mathbf{v}_r}{c} \wedge \mathbf{H} \right) + 2\lambda \sum_a \frac{\partial H_a}{\partial \mathbf{x}} \cdot (\Psi^* \Sigma_{a,\text{op}} \Psi). \end{aligned}$$

By taking into account (25), (26) and (27), we get from (24)

$$(28) \quad m \sum_r R_r^2 \left( \frac{\partial}{\partial t} + \mathbf{v}_r \cdot \frac{\partial}{\partial \mathbf{x}} \right) \mathbf{v}_r = e \sum_r R_r^2 \left( \mathbf{E} + \frac{\mathbf{v}_r}{c} \wedge \mathbf{H} \right) + \\ + \lambda \sum_a \frac{\partial H_a}{\partial \mathbf{x}} \cdot (\Psi^* \boldsymbol{\Sigma}_{\text{op}} \Psi)_a + \frac{\hbar^2}{2m} \sum_r R_r^2 \frac{\partial}{\partial \mathbf{x}} \frac{\Delta R_r}{R_r} - m \sum_r \left( \frac{\partial R_r^2}{\partial t} + \text{div} (R_r^2 \mathbf{v}_r) \right) \mathbf{v}_r.$$

In the case of a spinless particle, equation (28) goes over into the Euler equation (5) of section 1, because the second and fourth terms in the right hand side vanish. The form of equation (28) shows that, in the case of a spinning particle, the Madelung fluid behaves as a mixture of  $2s+1$  phases having different densities  $mR_r^2$  and different velocities  $\mathbf{v}_r$ . This is seen in a clearer way by splitting (28) into a set of  $2s+1$  equations

$$(29) \quad m \left( \frac{\partial}{\partial t} + \mathbf{v}_r \cdot \frac{\partial}{\partial \mathbf{x}} \right) \mathbf{v}_r = e \left( \mathbf{E} + \frac{\mathbf{v}_r}{c} \wedge \mathbf{H} \right) + \frac{\hbar^2}{2m} \frac{\partial}{\partial \mathbf{x}} \frac{\Delta R_r}{R_r} + \\ + \frac{\partial}{\partial \mathbf{x}} \sum_{r'} \frac{\lambda}{2R_r^2} (\Psi_r^* \langle r' | \mathbf{H} \cdot \boldsymbol{\Sigma}_{\text{op}} | r \rangle \Psi_r + \Psi_r^* \langle r | \mathbf{H} \cdot \boldsymbol{\Sigma}_{\text{op}} | r' \rangle \Psi_{r'}) .$$

In order to prove equation (29), we shall replace the Schrödinger equations for the  $\Psi_r$

$$(30) \quad i\hbar \frac{\partial \Psi_r}{\partial t} = \frac{1}{2m} \left( \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right)^2 \Psi_r + eA_0 \Psi_r - \lambda \sum_{r'} \langle r | \mathbf{H} \cdot \boldsymbol{\Sigma}_{\text{op}} | r' \rangle \Psi_{r'} ,$$

by equations for the  $R_r$  and  $S_r$

$$(31) \quad \frac{\partial S_r}{\partial t} + \frac{1}{2m} \left( \frac{\partial S_r}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right)^2 + eA_0 - \frac{\hbar^2}{2m} \frac{\Delta R_r}{R_r} - \\ - \frac{\lambda}{2R_r^2} \sum_{r'} (\Psi_r^* \langle r' | \mathbf{H} \cdot \boldsymbol{\Sigma}_{\text{op}} | r \rangle \Psi_r + \Psi_r^* \langle r | \mathbf{H} \cdot \boldsymbol{\Sigma}_{\text{op}} | r' \rangle \Psi_{r'}) = 0 ,$$

$$(32) \quad \frac{\partial R_r^2}{\partial t} + \text{div} (R_r^2 \mathbf{v}_r) = \frac{i\lambda}{\hbar} \sum_{r'} (\Psi_r^* \langle r | \mathbf{H} \cdot \boldsymbol{\Sigma}_{\text{op}} | r' \rangle \Psi_{r'} - \Psi_{r'}^* \langle r' | \mathbf{H} \cdot \boldsymbol{\Sigma}_{\text{op}} | r \rangle \Psi_r) .$$

By taking the gradients of both sides of (31) and using the well known identity

$$(33) \quad \frac{1}{2} \frac{\partial}{\partial \mathbf{x}} u^2 = \left( \mathbf{u} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \mathbf{u} + \mathbf{u} \wedge \text{rot} \mathbf{u} ,$$

we get equation (29).

Equation (29) shows that each of the  $2s+1$  phases moves under the action of a different quantum potential  $U_r$

$$(34) \quad U_r = -\frac{\hbar^2}{2m} \frac{\Delta R_r}{R_r} - \frac{\lambda}{2R_r^2} \sum_{r'} (\Psi_{r'}^* \langle r' | \mathbf{H} \cdot \boldsymbol{\Sigma}_{op} | r \rangle \Psi_r + \Psi_r^* \langle r | \mathbf{H} \cdot \boldsymbol{\Sigma}_{op} | r' \rangle \Psi_{r'}) .$$

It follows from equation (32) that the mass of fluid contained in any phase is not conserved, as should be expected. Those phases have no invariant meaning, since they depend on the choice of the axes of coordinates.

Equation (4) shows that, in the absence of magnetic fields, the motion of each of the phases is irrotational. The velocity  $\mathbf{v}(t, \mathbf{x})$  of the Madelung fluid defined by (15) is the average of the velocities of the  $2s+1$  phases (center of mass velocity)

$$(35) \quad \mathbf{v}(t, \mathbf{x}) = \sum_r R_r^2 \mathbf{v}_r / \sum_r R_r^2 = \sum_r c_r \mathbf{v}_r, \quad \left( c_r = \frac{R_r^2}{\sum_{r'} R_{r'}^2} \right),$$

$$(36) \quad \text{rot } \mathbf{v} = \sum_r c_r \text{rot } \mathbf{v}_r + \sum_r \frac{\partial \varphi_r}{\partial \mathbf{x}} \wedge \mathbf{v}_r,$$

*rot v may be different from 0, even in the absence of magnetic fields.*

It follows from (25) and (23) that

$$(37) \quad \frac{\partial}{\partial t} \langle G_{a,op} \rangle_{\Psi} + m \sum_{b=1}^3 \frac{\partial}{\partial x_b} (\varrho_M v_a v_b) = m \varrho_M \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) v_a,$$

hence the Euler equations for the center of mass velocity are

$$(38) \quad m \varrho_M \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) v_a = - \sum_{b=1}^3 \frac{\partial}{\partial x_b} \Theta_{ab} + c \sum_r R_r^2 \left( \mathbf{E} + \frac{\mathbf{v}_r}{c} \wedge \mathbf{H} \right) + \\ + \lambda \sum_{b=1}^3 \frac{\partial H_b}{\partial x_a} (\Psi^* \boldsymbol{\Sigma}_{op} \Psi)_b,$$

$\Theta_{ab}$  is the elastic stress tensor

$$(39) \quad \Theta_{ab} = \langle T_{ab}(t, \mathbf{x})_{op} \rangle_{\Psi} - m \varrho_M v_a v_b = \\ = m \sum_r R_r^2 (v_{r,a} - v_a)(v_{r,b} - v_b) + \frac{\hbar^2}{m} \sum_r \frac{\partial R_r}{\partial x_a} \frac{\partial R_r}{\partial x_b} - \frac{\hbar^2}{4m} \delta_{ab} \Delta (\sum_r R_r^2).$$

The Reynolds terms arising from the turbulence are now

$$(40) \quad m \sum_r R_r^2 (v_{r,a} - v_a)(v_{r,b} - v_b) + \frac{\hbar^2}{m} \sum_r \frac{\partial R_r}{\partial x_a} \frac{\partial R_r}{\partial x_b}.$$

By means of simple transformations, we get from (38)

$$(41) \quad m \varrho_M \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \mathbf{v} = e \varrho_M \left( \mathbf{E} + \frac{\mathbf{v}}{c} \wedge \mathbf{H} \right) + \lambda \sum_{b=1}^3 \frac{\partial H_b}{\partial \mathbf{x}} \cdot (\Psi^* \Sigma_{op} \Psi)_b + \\ + \frac{\hbar^2}{2m} \sum_r R_r^2 \frac{\partial}{\partial \mathbf{x}} \frac{\Delta R_r}{R_r} - m \sum_{b=1}^3 \frac{\partial}{\partial x_b} \sum_r R_r^2 (v_{r,b} - v_r)(\mathbf{v}_r - \mathbf{v}).$$

It is easily seen that:

$$(42) \quad \frac{d}{dt} \Sigma(t)_{op}^H = \lambda \Sigma(t)_{op}^H \wedge \mathbf{H}(t, \mathbf{x}(t)_{op}^H).$$

The equation (26) of section 1 follows immediately from (42). By taking the expectation values of both sides of equation (26) of section 1, we get

$$(43) \quad \frac{\partial}{\partial t} (\varrho_M \Sigma_a) + \operatorname{div} \langle \Sigma_{a,op} \mathbf{j}(t, \mathbf{x})_{op} \rangle_\Psi = \lambda \varrho_M (\Sigma \wedge \mathbf{H})_a,$$

Since

$$(44) \quad \frac{\partial}{\partial t} (\varrho_M \Sigma_a) = \varrho_M \frac{\partial}{\partial t} \Sigma_a - \Sigma_a \operatorname{div} (\varrho_M \mathbf{v}) = \\ = \varrho_M \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \Sigma_a - \operatorname{div} (\varrho_M \mathbf{v} \Sigma_a),$$

we get the equation (28) of section 1.

It follows from (43) that

$$(45) \quad \frac{\partial}{\partial t} (\varrho_M \Sigma_a) + \operatorname{div} (\Sigma_a \mathbf{j}(t, \mathbf{x})) = \lambda \varrho_M (\mathbf{E} \wedge \mathbf{H})_a + \\ + \operatorname{div} (\Sigma_a \mathbf{j}(t, \mathbf{x}) - \langle \Sigma_{a,op} \mathbf{j}(t, \mathbf{x})_{op} \rangle_\Psi).$$

The existence of the second term in the right hand side of (45) means that the variation of the spin vector  $\Sigma(t, \mathbf{x})$  is not due entirely to the electromagnetic torque  $\lambda \Sigma \wedge \mathbf{H}$ . There is a quantum torque analogous to the force derived from the quantum potential.

### 3. - Quantization of the motions of the Madelung fluid.

The equations of the quasi-irrotational motions of the spinless Madelung fluid follow from the variational principle

$$(1) \quad \delta \int R^2 \left\{ \frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right)^2 + e A_0 + U \right\} dt d\mathbf{x} = 0.$$

It is convenient to take as density of the lagrangian the function  $L$

$$(2) \quad L = - \left\{ \frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right)^2 + eA_0 + U \right\} R^2.$$

Thus we get a satisfactory expression for the energy

$$(3) \quad \mathcal{H} = \int R^2 \left\{ \frac{1}{2m} \left( \frac{\partial S}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right)^2 + eA_0 + U \right\} d\mathbf{x}.$$

The conjugate momentum of  $S$  is  $-R^2$ . It is well known that, for the quantization, the variables  $S$  and  $-R^2$  are not satisfactory, because of the non existence of operators  $S(\mathbf{x})_{\text{op}}$ ,  $R(\mathbf{x})_{\text{op}}$  fulfilling the commutation rule

$$(4) \quad [R^2(\mathbf{x})_{\text{op}}, S(\mathbf{x}')_{\text{op}}] = i\hbar \delta(\mathbf{x} - \mathbf{x}').$$

In order to perform the quantization, we shall introduce the variables  $\Psi$ ,  $i\hbar\Psi^*$  and the corresponding operators  $\psi(\mathbf{x})_{\text{op}}$  and  $i\hbar\psi^*(\mathbf{x})_{\text{op}}$ , satisfying the commutation rules

$$(5) \quad [\psi(\mathbf{x})_{\text{op}}, \psi^*(\mathbf{x}')_{\text{op}}] = \delta(\mathbf{x} - \mathbf{x}'), \quad [\psi(\mathbf{x})_{\text{op}}, \psi(\mathbf{x}')_{\text{op}}] = 0.$$

Since

$$(6) \quad \mathcal{H} = \int \Psi^* \left\{ \frac{1}{2m} \left( \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right)^2 + eA_0 \right\} \Psi d\mathbf{x} = \int \Psi^* H_{\text{op}} \Psi d\mathbf{x},$$

the hamiltonian operator is  $\mathcal{H}_{\text{op}}$

$$(7) \quad \mathcal{H}_{\text{op}} = \int_{-\infty}^{+\infty} \psi^*(\mathbf{x})_{\text{op}} \left\{ \frac{1}{2m} \left( \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right)^2 + eA_0 \right\} \psi(\mathbf{x})_{\text{op}} d\mathbf{x}.$$

$\mathcal{H}_{\text{op}}$  coincides with the hamiltonian of the second quantization of the Schrödinger equation, for non interacting particles.

The commutation rules (5) correspond to the general quantization procedure, because the variational principle (1) is equivalent to the Schrödinger principle

$$(8) \quad \delta \int \Psi^* \left\{ i\hbar \frac{\partial}{\partial t} - \frac{1}{2m} \left( \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right)^2 - eA_0 \right\} \Psi dt d\mathbf{x} = 0.$$



and  $i\hbar\Psi^*$  is the conjugate momentum of  $\Psi$ . The rules (5) lead to the Bose statistics, which is the right choice for spinless particles. It is nevertheless interesting to consider also the commutation rules for the Fermi statistics

$$(9) \quad [\psi(\mathbf{x})_{\text{op}}, \psi^*(\mathbf{x}')_{\text{op}}]_{\pm} = \delta(\mathbf{x} - \mathbf{x}'), \quad [\psi(\mathbf{x})_{\text{op}}, \psi(\mathbf{x}')_{\text{op}}]_{\pm} = 0,$$

because the following results are valid for both statistics.

The second quantization operator  $\mathcal{K}_{\text{op}}$ , that corresponds to the one particle quantity represented by the operator  $K_{\text{op}}$ , is given by the formula

$$(10) \quad \mathcal{K}_{\text{op}} = \int_{-\infty}^{+\infty} \psi^*(\mathbf{x})_{\text{op}} K_{\text{op}} \psi(\mathbf{x})_{\text{op}} d\mathbf{x}.$$

The formulae (44) of section 1 were obtained by the application of (10). The operator for the charge density being  $e\varrho^{(2)}(\mathbf{x})_{\text{op}}$ , that for the total number of particles must be  $N_{\text{op}}$

$$(11) \quad eN_{\text{op}} = e \int_{-\infty}^{+\infty} \varrho^{(2)}(\mathbf{x})_{\text{op}} d\mathbf{x} = e \int_{-\infty}^{+\infty} \psi^*(\mathbf{x})_{\text{op}} \psi(\mathbf{x})_{\text{op}} d\mathbf{x}.$$

It is well known that  $N_{\text{op}}$  has the eigenvalue 0, 1, 2, .... The vacuum is the state in which the total number of particles is 0. We shall denote the normalized wave functional of the vacuum by  $\chi_0$  and the variables of the fluid by  $\omega$

$$(12) \quad N_{\text{op}}\chi_0 = 0, \quad \int |\chi_0|^2 d\omega = 0.$$

The operator for the total number of particles in the one particle state described by the normalized wave function  $\Phi$  is  $N_{\Phi}$

$$(13) \quad N_{\Phi} = a_{\Phi}^* a_{\Phi}, \quad a_{\Phi} = \int_{-\infty}^{+\infty} \Phi^*(\mathbf{x}) \psi(\mathbf{x})_{\text{op}} d\mathbf{x}.$$

We get from (5) and (9):

$$(14) \quad [a_{\Phi}, a_{\Phi}^*]_{\pm} = \int_{-\infty}^{+\infty} \Phi^*(\mathbf{x}) \Phi'(\mathbf{x}) d\mathbf{x}, \quad [a_{\Phi}, a_{\Phi}]_{\pm} = 0.$$

The eigenvalues of  $N_{\Phi}$  are 0 and 1, in the case of the fermions, and the non negative integers, in the case of bosons. Let us introduce the wave func-

tional  $\chi_\Phi$

$$(16) \quad \chi_\Phi = \int_{-\infty}^{+\infty} \Phi(\mathbf{x}) \psi^*(\mathbf{x})_{\text{op}} \chi_0 \, d\mathbf{x} = a_\Phi^* \chi_0 .$$

Since  $\psi_{\text{op}} \chi_0 = 0$ , we have

$$(17) \quad \psi(\mathbf{x})_{\text{op}} \chi_\Phi = \Phi(\mathbf{x}) \chi_0, \quad a_\Phi \chi_\Phi = \int_{-\infty}^{+\infty} \Phi'^* \Phi \, d\mathbf{x} \, \chi_0 .$$

Hence

$$(18) \quad N_{\Phi'} \chi_\Phi = \int_{-\infty}^{+\infty} \Phi'^* \Phi \, d\mathbf{x} \, \chi_{\Phi'}, \quad N_\Phi \chi_\Phi = \chi_\Phi,$$

so that there is one quantum in the particle state  $\Phi$  and no quanta in the particle states  $\Phi'$  orthogonal to  $\Phi$ , when the fluid is in the state  $\chi_\Phi$ . It follows from the first equation (17) that the total number of quanta in the fluid state  $\chi_\Phi$  is 1

$$(19) \quad N_{\text{op}} \chi_\Phi = \chi_\Phi$$

and also that

$$(20) \quad \mathcal{R}_{\text{op}} \chi_\Phi = \int_{-\infty}^{+\infty} \psi^*(\mathbf{x})_{\text{op}} K_{\text{op}} \Phi(\mathbf{x}) \chi_0 \, d\mathbf{x} .$$

Hence

$$(21) \quad \begin{aligned} \int \chi_\Phi^* \mathcal{R}_{\text{op}} \chi_\Phi \, d\omega &= \int d\omega \chi_\Phi^* \int_{-\infty}^{+\infty} \psi^*(\mathbf{x})_{\text{op}} K_{\text{op}} \Phi(\mathbf{x}) \chi_0 \, d\mathbf{x} = \\ &= \int d\omega \chi_0^* \int_{-\infty}^{+\infty} \Phi^*(\mathbf{x}') \, d\mathbf{x}' \int_{-\infty}^{+\infty} \psi(\mathbf{x}')_{\text{op}} \psi^*(\mathbf{x})_{\text{op}} K_{\text{op}} \Phi(\mathbf{x}) \chi_0 \, d\mathbf{x} = \\ &= \int \chi_0^* \chi_0 \, d\omega \cdot \int_{-\infty}^{+\infty} \Phi^*(\mathbf{x}) K_{\text{op}} \Phi(\mathbf{x}) \, d\mathbf{x} = \int_{-\infty}^{+\infty} \Phi^* K_{\text{op}} \Phi \, d\mathbf{x} , \end{aligned}$$

and by taking  $K_{op} = 1$  we get

$$(22) \quad \int \chi_{\Phi}^* \chi_{\Phi} d\omega = 1, \quad \left( \int_{-\infty}^{+\infty} |\Phi|^2 d\mathbf{x} = 1 \right).$$

Equation (22) shows that  $\chi_{\Phi}^*$  is normalized. Therefore  $\int \chi_{\Phi}^* \mathcal{R}_{op} \chi_{\Phi} d\omega$  is the expectation value of the quantity  $\mathcal{R}_{op}$  in the state  $\chi_{\Phi}$  of the fluid. Equation (21) shows that

$$(23) \quad \langle \mathcal{R}_{op} \rangle_{\chi_{\Phi}} = \langle K_{op} \rangle_{\Phi}.$$

It follows from (23) that the probabilities of the values of the physical quantities in the state  $\chi_{\Phi}$  of the quantized Madelung fluid are the same given by the ordinary interpretation rules of the quantum mechanics for a particle in the state  $\Phi$ , because the expectation values are the same and the possible values of the fluid quantity  $\mathcal{R}_{op}$  in the one quantum states are the same as those of  $K_{op}$ , i.e. the eigenvalues of  $K_{op}$ , as we shall now prove. In order that  $\chi_{\Phi}$  be an eigenfunctional of  $\mathcal{R}_{op}$  corresponding to the eigenvalue  $\mathcal{R}'$ , it is necessary and sufficient that

$$(24) \quad \int_{-\infty}^{+\infty} \psi^*(\mathbf{x}')_{op} K_{op} \Phi'(\mathbf{x}') \chi_0 d\mathbf{x}' = \mathcal{R}' \int_{-\infty}^{+\infty} \psi^*(\mathbf{x}')_{op} \Phi'(\mathbf{x}') \chi_0 d\mathbf{x}',$$

as a consequence of (20). By the application of  $\psi(\mathbf{x})_{op}$  to both sides of (24), we see that  $\Phi'$  must be an eigenfunction of  $K_{op}$ ,  $\mathcal{R}'$  being the corresponding eigenvalue

$$(25) \quad \psi(\mathbf{x})_{op} \int_{-\infty}^{+\infty} \psi^*(\mathbf{x}')_{op} \chi_0 (K_{op} - \mathcal{R}') \Phi'(\mathbf{x}') d\mathbf{x}' = \chi_0 (K_{op} - \mathcal{R}') \Phi(\mathbf{x}) = 0.$$

Equation (24) shows that to any eigenfunction  $\Phi'$  of  $K_{op}$  corresponds an eigenfunctional  $\chi_{\Phi'}$  of  $\mathcal{R}_{op}$ , the eigenvalues being the same.

Let us consider the operator  $N(V)_{op}$  for the number of particles within the volume  $V$

$$(26) \quad N(V)_{op} = \int_V \psi^*(\mathbf{x})_{op} \psi(\mathbf{x})_{op} d\mathbf{x}.$$

It is well known that the eigenvalues of  $N(V)_{op}$  are the non negative integers. In the case of the one quantum states, the possible values of the number of

particles within  $V$  can only be 0 or 1. This is shown directly by the equation

$$(27) \quad N(V)_{op} \chi_\Phi = \int_V \Phi(\mathbf{x}) \psi^*(\mathbf{x})_{op} \chi_0 d\mathbf{x}.$$

The eigenvalue 1 corresponds to the  $\Phi$  that vanish outside  $V$ , and the eigenvalue 0 to those vanishing inside  $V$ . The expectation value of  $N(V)_{op}$  in the state  $\chi_\Phi$  being

$$(28) \quad \langle N(V)_{op} \rangle_{\chi_\Phi} = \int d\omega \chi_0^* \int_{-\infty}^{+\infty} \psi(\mathbf{x})_{op} \Phi^*(\mathbf{x}) d\mathbf{x} \int_V \Phi(\mathbf{x}') \psi^*(\mathbf{x}')_{op} \chi_0 d\mathbf{x}' = \\ = \int_V |\Phi|^2 d\mathbf{x},$$

the probability of the eigenvalue 1 is also  $\int_V |\Phi|^2 d\mathbf{x}$ , so that we get as the probability of finding the particle inside  $V$  the mass of the non quantized fluid inside  $V$ , divided by  $m$ . This result shows clearly how the quantization of the Madelung fluid allows to get the ordinary interpretation of the Schrödinger equation. *The quantization of the Madelung fluid regenerates the turbulence that was eliminated by taking the expectation values in the one particle formalism of section 2.*

In the Heisenberg representation of the quantized fluid, we have time dependent absorption and emission operators  $\psi(t, \mathbf{x})_{op}^H$  and  $\psi^*(t, \mathbf{x})_{op}^H$ . It follows from the Heisenberg equation of motion that  $\psi(t, \mathbf{x})_{op}^H$  satisfies the Schrödinger equation

$$(29) \quad \left( i\hbar \frac{\partial}{\partial t} - \frac{1}{2m} \left( \hbar \frac{\partial}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right)^2 - eA_0 \right) \psi(t, \mathbf{x})_{op}^H = 0.$$

Since

$$(30) \quad \mathcal{R}(t)_{op}^H = \int_{-\infty}^{+\infty} \psi^*(t, \mathbf{x})_{op}^H K_{op} \psi(t, \mathbf{x})_{op}^H d\mathbf{x},$$

we get from (29)

$$(31) \quad \mathcal{R}(t)_{op}^H = \int_{-\infty}^{+\infty} \psi^*(\mathbf{x})_{op} K(t)_{op}^H \psi(\mathbf{x})_{op} d\mathbf{x}.$$

It follows from the continuity equation (19) of section 1 and the equation (31) that the operators for the charge and current densities of the quant-

ized fluid satisfy the following continuity equation

$$\frac{\partial}{\partial t} \varrho^{(2)}(t, \mathbf{x})_{\text{op}}^H + \operatorname{div} \mathbf{j}^{(2)}(t, \mathbf{x})_{\text{op}}^H = 0.$$

In a similar way we get from the equation (20) of section 1

$$(33) \quad \frac{\partial}{\partial t} G_a^{(2)}(t, \mathbf{x})_{\text{op}}^H + \sum_{b=1}^3 \frac{\partial}{\partial x_b} T_{ab}^{(2)}(t, \mathbf{x})_{\text{op}}^H = \frac{1}{2} \int_{-\infty}^{+\infty} \psi^*(t, \mathbf{x})_{\text{op}} \{f_a(t)_{\text{op}}, \varrho(\mathbf{x})_{\text{op}}\} \psi(t, \mathbf{x})_{\text{op}} d\mathbf{x} = \\ = e \varrho^{(2)}(t, \mathbf{x})_{\text{op}}^H E_a(t, \mathbf{x}) + \frac{e}{c} (\mathbf{j}^{(2)}(t, \mathbf{x})_{\text{op}}^H \wedge \mathbf{H}(t, \mathbf{x}))_a,$$

with

$$(34) \quad \mathbf{G}^{(2)}(t, \mathbf{x})_{\text{op}} = \frac{1}{2} \left( \psi^*(\mathbf{x})_{\text{op}} \left( \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} - \frac{e}{c} \mathbf{A} \right) \psi(\mathbf{x})_{\text{op}} - \right. \\ \left. - \left\{ \left( \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}} + \frac{e}{c} \mathbf{A} \right) \psi^*(\mathbf{x})_{\text{op}} \right\} \psi(\mathbf{x})_{\text{op}} \right).$$

The quantization of the Madelung fluid for particles with spin can be performed without difficulty, since it corresponds to the second quantization of their Schrödinger equations, for non interacting particles. The commutation rules are given by equations (42) of section 1. The densities of charge and current and the components of the stress tensor (spatial) are defined by equations (43). The only difference lies in the fact that now there is one absorption operator for each component of the wave spinor, so that  $\psi(\mathbf{x})_{\text{op}}$  is a one column matrix of operators and  $\psi^*(\mathbf{x})_{\text{op}}$  a one row matrix. Whereas in the density formalism of section 2 the spin of the particle did not affect the definition of the operator densities of charge, current, mass and momentum and that of the spatial components of the stress tensor, this is no more true in the theory of the quantized fluid. Thus we have

$$(35) \quad \varrho^{(2)}(\mathbf{x}) = \sum_r \psi_r^*(\mathbf{x})_{\text{op}} \psi_r(\mathbf{x})_{\text{op}}.$$

In the case of a particle with magnetic moment, equation (33) is replaced by the following equation

$$(36) \quad \frac{\partial}{\partial t} G_a^{(2)}(t, \mathbf{x})_{\text{op}}^H + \sum_{b=1}^3 \frac{\partial}{\partial x_b} T_{ab}^{(2)}(t, \mathbf{x})_{\text{op}}^H = e \varrho^{(2)}(t, \mathbf{x})_{\text{op}}^H E_a(t, \mathbf{x}) + \\ + \frac{e}{c} (\mathbf{j}^{(2)}(t, \mathbf{x})_{\text{op}}^H \wedge \mathbf{H}(t, \mathbf{x}))_a + \lambda \Sigma^{(2)}(t, \mathbf{x})_{\text{op}}^H \cdot \frac{\partial \mathbf{H}}{\partial x_a},$$

$$(37) \quad \Sigma^{(2)}(t, \mathbf{x})_{\text{op}}^H = \psi^*(t, \mathbf{x})_{\text{op}}^H \Sigma_{\text{op}} \psi(t, \mathbf{x})_{\text{op}}^H = \int_{-\infty}^{+\infty} \psi^*(t, \mathbf{x}')_{\text{op}}^H \boldsymbol{\sigma}(\mathbf{x})_{\text{op}} \psi(t, \mathbf{x}')_{\text{op}}^H d\mathbf{x}'.$$



It follows from (42) of section 2 that

$$(38) \quad \frac{\partial}{\partial t} \Sigma_a^{(2)}(t, \mathbf{x})_{\text{op}}^H + \text{div} \mathbf{\Omega}_a(t, \mathbf{x})_{\text{op}}^H = \lambda (\Sigma^{(2)}(t, \mathbf{x})_{\text{op}}^H \wedge \mathbf{H}(t, \mathbf{x}))_a,$$

$$(39) \quad \mathbf{\Omega}_a(t, \mathbf{x})_{\text{op}} = \frac{1}{2} \psi^*(\mathbf{x})_{\text{op}} \Sigma_{a,\text{op}} \dot{\mathbf{x}}(t)_{\text{op}} \psi(\mathbf{x})_{\text{op}} + \frac{1}{2} (\dot{\mathbf{x}}(t)_{\text{op}} \psi(\mathbf{x})_{\text{op}})^* \Sigma_{a,\text{op}} \psi(\mathbf{x})_{\text{op}}.$$

#### 4. - The fluctuations of the quantized Madelung fluid.

We have seen, in the preceding section, that the one quantum states of the quantized Madelung fluid have the same properties of turbulence as the medium associated to the ordinary interpretation of the one particle Schrödinger equation. The quantization of the fluid introduces fluctuations that give rise to the turbulence. We shall now see that these fluctuations are closely related to the zero-point fluctuations of the vacuum.

The operators that do not commute with  $N_{\text{op}}$  do not, in general, admit  $\chi_0$  as an eigenfunctional and do not have definite values in the state  $\chi_0$  of the fluid. The corresponding fluid variables « fluctuate » in the vacuum. The operators  $\psi(\mathbf{x})_{\text{op}}$  and  $\psi^*(\mathbf{x})_{\text{op}}$  do not commute with  $N_{\text{op}}$

$$(1) \quad [\psi(\mathbf{x})_{\text{op}}, N_{\text{op}}] = \psi(\mathbf{x})_{\text{op}}, \quad [\psi^*(\mathbf{x})_{\text{op}}, N_{\text{op}}] = -\psi^*(\mathbf{x})_{\text{op}}.$$

The above commutation rules characterize  $\psi(\mathbf{x})_{\text{op}}$  as an absorption operator and  $\psi^*(\mathbf{x})_{\text{op}}$  as an emission operator. Although  $\psi(\mathbf{x})_{\text{op}}$  does not commute with  $N_{\text{op}}$ , it admits  $\chi_0$  as an eigenfunctional corresponding to the eigenvalue 0.  $\chi_0$  is no eigenfunctional of  $\psi^*(\mathbf{x})_{\text{op}}$ , since  $\psi^*(\mathbf{x})_{\text{op}}\chi_0$  is a one quantum state of the fluid, in which the particle is localized at the point  $\mathbf{x}$ . Indeed, we have

$$(2) \quad \psi^*(\mathbf{x})_{\text{op}}\chi_0 = \int_{-\infty}^{+\infty} \delta(\mathbf{x} - \mathbf{x}') \psi_{\text{op}}^*(\mathbf{x}') \chi_0 d\mathbf{x}',$$

and it follows from equation (27) of section 3 that

$$(3) \quad N(V)_{\text{op}} \psi^*(\mathbf{x})_{\text{op}} \chi_0 = \int_V \delta(\mathbf{x} - \mathbf{x}') \psi_{\text{op}}^*(\mathbf{x}') \chi_0 d\mathbf{x}'$$

$$(4) \quad N(V)_{\text{op}} \psi^*(\mathbf{x})_{\text{op}} \chi_0 = \begin{cases} 0 & \text{point } \mathbf{x} \text{ outside } V. \\ \psi^*(\mathbf{x})_{\text{op}} \chi_0 & \text{point } \mathbf{x} \text{ inside } V. \end{cases}$$

The vacuum fluctuation of the fluid variable  $\psi^*(\mathbf{x})_{op}$  is thus connected with the creation of a quantum at the point  $\mathbf{x}$ .

The one quantum wave functionals  $\chi_\phi$  are formed by the superposition of the wave functionals  $\psi^*(\mathbf{x})_{op}\chi_0$  corresponding to all the points of space, as shown by the equation (16) of section 3.  $|\Phi|^2$  measures the contribution of  $\psi^*(\mathbf{x})_{op}\chi_0$  to the state  $\chi_\phi$  of the fluid. The  $n$ -quanta state of the fluid  $\chi_{\phi_n}$  are described by functionals of the form

$$(5) \quad \chi_{\phi_n} = \frac{1}{\sqrt{n!}} \int_{-\infty}^{+\infty} \Phi_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \psi^*(\mathbf{x}_1)_{op} \dots \psi^*(\mathbf{x}_n)_{op} \chi_0 d\mathbf{x}_1 \dots d\mathbf{x}_n.$$

The wave functionals are formed by the superposition of the  $\psi^*(\mathbf{x}_1)_{op} \dots \psi^*(\mathbf{x}_n)_{op} \chi_0$ , that are related with the vacuum fluctuation of the  $n$ -point fluid quantity described by  $\psi^*(\mathbf{x}_1)_{op} \dots \psi^*(\mathbf{x}_n)_{op}$ . These fluctuations are associated with the creation of a set of  $n$  particles localized at the point  $\mathbf{x}_1, \dots, \mathbf{x}_n$ :

The general states of the quantized fluid can be represented by series of the form

$$(6) \quad \chi = \Phi_0 \chi_0 + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \int_{-\infty}^{+\infty} \Phi_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \psi^*(\mathbf{x}_1)_{op} \dots \psi^*(\mathbf{x}_n)_{op} \chi_0 d\mathbf{x}_1 \dots d\mathbf{x}_n.$$

They can be associated with the vacuum fluctuation of the fluid quantity described by the operator  $\Phi_0 + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \int_{-\infty}^{+\infty} \Phi_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \psi^*(\mathbf{x}_1)_{op} \dots \psi^*(\mathbf{x}_n)_{op} d\mathbf{x}_1 \dots d\mathbf{x}_n$ .

In the case of the Bose statistics, we shall consider the important wave functional

$$(7) \quad \chi[\Phi] = \left( \exp \left[ \int_{-\infty}^{+\infty} \Phi(\mathbf{x}) \psi^*(\mathbf{x})_{op} d\mathbf{x} \right] \right) \chi_0.$$

Since

$$(8) \quad \left[ \psi(\mathbf{x})_{op}, \int_{-\infty}^{+\infty} \Phi(\mathbf{x}') \psi^*(\mathbf{x}')_{op} d\mathbf{x}' \right] = \Phi(\mathbf{x}),$$

we have

$$(9) \quad \left[ \psi(\mathbf{x})_{op}, \left( \int_{-\infty}^{+\infty} \Phi(\mathbf{x}') \psi^*(\mathbf{x}')_{op} d\mathbf{x}' \right)^n \right] = n \Phi(\mathbf{x}) \left( \int_{-\infty}^{+\infty} \Phi(\mathbf{x}') \psi^*(\mathbf{x}')_{op} d\mathbf{x}' \right)^{n-1}.$$

Hence

$$(10) \quad \left( \psi(\mathbf{x})_{\text{op}}, \exp \left[ \int_{-\infty}^{+\infty} \Phi(\mathbf{x}') \psi^*(\mathbf{x}')_{\text{op}} d\mathbf{x}' \right] \right) = \Phi(\mathbf{x}) \exp \left[ \int_{-\infty}^{+\infty} \Phi(\mathbf{x}') \psi^*(\mathbf{x}')_{\text{op}} d\mathbf{x}' \right],$$

and we see that  $\chi[\Phi]$  is an eigenfunctional of  $\psi(\mathbf{x})_{\text{op}}$  corresponding to the eigenvalue  $\Phi(\mathbf{x})$

$$(11) \quad \psi(\mathbf{x})_{\text{op}} \chi[\Phi] = \Phi(\mathbf{x}) \chi[\Phi].$$

We have seen, in section 3, that the values of the quantities in the state of the Madelung fluid described by the function  $\Phi(\mathbf{x})$  coincide with the expectation values of the corresponding operators for the quantized fluid in the state  $\chi_\Phi$ . We may consider the expectation values of those operators in any state  $\chi$  of the quantized fluid. By taking the expectation values in any state of the quantized fluid we will not get back to the simple unquantized Madelung fluid, but to some generalization. We shall see that we can get in that way ensembles of fluids, so that the present generalization includes the statistical mechanics of the unquantized Madelung fluid.

We shall need the following formula for the integration with respect to the variables involved in the wave functionals of the quantized fluid

$$(12) \quad \int \chi^* \chi' d\omega = \Phi_0^* \Phi'_0 + \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} \Phi_n^*(\mathbf{x}_1, \dots, \mathbf{x}_n) \Phi'_n(\mathbf{x}_1, \dots, \mathbf{x}_n) d\mathbf{x}_1 \dots d\mathbf{x}_n.$$

$\chi$  and  $\chi'$  being two functionals of the type (6), whose coefficients are the functions  $\Phi_n(\mathbf{x}_1, \dots, \mathbf{x}_n)$  and  $\Phi'_n(\mathbf{x}_1, \dots, \mathbf{x}_n)$ . These functions must be taken either as symmetrical, in the case of the Bose statistics, or anti-symmetrical, in the case of the Fermi statistics. It follows from (12) that the normalization condition for a functional (6) is

$$(13) \quad |\Phi_0|^2 + \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} |\Phi_n|^2 d\mathbf{x}_1 \dots d\mathbf{x}_n = 1.$$

It is easily seen that

$$(14) \quad \mathcal{R}_{\text{op}} \chi = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \int_{-\infty}^{+\infty} \psi^*(\mathbf{x}_1)_{\text{op}} \dots \psi^*(\mathbf{x}_n)_{\text{op}} \sum_{l=1}^n K_{\text{op}}(\mathbf{x}_l) \Phi_n(\mathbf{x}_1, \dots, \mathbf{x}_n) d\mathbf{x}_1 \dots d\mathbf{x}_n.$$

We get from (14) and (12)

$$(15) \quad \int \chi^* \mathcal{R}_{op} \chi d\omega = \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} \Phi_n^*(\mathbf{x}_1, \dots, \mathbf{x}_n) \sum_{l=1}^n K_{op}(\mathbf{x}_l) \Phi_n(\mathbf{x}_1, \dots, \mathbf{x}_n) d\mathbf{x}_1 \dots d\mathbf{x}_n.$$

In the simple case of a state  $\chi_n$ , in which there are  $n$  quanta in the orthogonal particle states  $\Phi^i(\mathbf{x})$

$$(16) \quad \chi_n = \frac{1}{\sqrt{n!}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{n!}} \sum_{\text{perm. } k} \pm \Phi^{k_1}(\mathbf{x}_1) \dots \Phi^{k_n}(\mathbf{x}_n) \psi^*(\mathbf{x}_1)_{op} \dots \psi^*(\mathbf{x}_n)_{op} \chi_0 d\mathbf{x}_1 \dots d\mathbf{x}_n,$$

$\chi_n$  will be normalized by taking normalized particle wave functions  $\Phi^i(\mathbf{x})$ . It follows from (15) that the expectation value of  $\mathcal{R}_{op}$  in the fluid state  $\chi_n$  is

$$(17) \quad \langle \mathcal{R}_{op} \rangle_{\chi_n} = \sum_{l=1}^n \int_{-\infty}^{+\infty} \Phi^{l*}(\mathbf{x}) K_{op} \Phi^l(\mathbf{x}) d\mathbf{x}.$$

By applying the formula (17) to the operators  $\varrho^{(2)}(\mathbf{x})_{op}$  and  $\mathbf{j}^{(2)}(\mathbf{x})_{op}$  we see that the charge density of the medium corresponding to the mean motion of the quantized Madelung fluid in the state  $\chi_n$  is the sum of the densities of  $n$  non interacting unquantized Madelung fluids in the states  $\Phi^i$ , the current density of the medium being also the sum of the current densities of the  $n$  unquantized fluids. The medium behaves as a mixture of  $n$  unquantized non interacting Madelung fluids, i.e. as an ensemble of  $n$  fluids. It is easily seen that each of these  $n$  fluids satisfies the equations of motion of the Madelung fluid.

We shall now discuss the expectation values in the state  $\chi[\Phi]$  defined by equation (7). The results we shall obtain have already been derived by us in a different way, <sup>(8)</sup> using the representation in which the operators  $\psi^*(\mathbf{x})_{op}$  are diagonalized. It follows from (12) that the wave functional  $\chi[\Phi]$  is not normalized

$$(18) \quad \int |\chi[\Phi]|^2 d\omega = \exp \left[ \int_{-\infty}^{+\infty} |\Phi|^2 d\mathbf{x} \right].$$

By the application of (15) we get

$$(19) \quad \int [\chi^*[\Phi] \mathcal{R}_{op} \chi[\Phi]] d\omega = \int_{-\infty}^{+\infty} \Phi^*(\mathbf{x}) K_{op} \Phi(\mathbf{x}) d\mathbf{x} \int |\chi[\Phi]|^2 d\omega,$$

<sup>(8)</sup> M. SCHÖNBERG: *Nuovo Cimento*, **10**, 1499 (1953).

so that

$$(20) \quad \langle \mathcal{R}_{op} \rangle_{\chi[\Phi]} = \int_{-\infty}^{+\infty} \Phi^*(\mathbf{x}) K_{op} \Phi(\mathbf{x}) d\mathbf{x}.$$

With a normalized  $\Phi$ , the expectation values of the  $\mathcal{R}_{op}$  in the state  $\chi[\Phi]$  of the fluid coincide with those of the  $K_{op}$  in the state  $\Phi$  of the particle

$$(21) \quad \langle \mathcal{R}_{op} \rangle_{\chi[\Phi]} = \langle K_{op} \rangle_{\Phi}.$$

*The expectation values of the quantities in the state  $\chi[\Phi]$  of the quantized fluid coincide with their values in the state  $\Phi$  of the Madelung fluid.*

Although the turbulence of the quantized fluid be different in the states  $\chi[\Phi]$  and  $\chi_{\Phi}$ , the corresponding mean motions are the same, provided  $\Phi$  be normalized. The functional  $\chi[\Psi(t, \mathbf{x})]$  satisfies the Schrödinger equation of the quantized fluid

$$(22) \quad i\hbar \frac{\partial}{\partial t} \chi[\Psi(t, \mathbf{x})] = \mathcal{H}_{op} \chi[\Psi(t, \mathbf{x})],$$

when  $\Psi(t, \mathbf{x})$  is a solution of the Schrödinger equation for the quantum, (1) of section 1. Indeed, since

$$(23) \quad i\hbar \frac{\partial}{\partial t} \exp \left[ \int_{-\infty}^{+\infty} \psi^*(\mathbf{x})_{op} \Psi(t, \mathbf{x}) d\mathbf{x} \right] = \\ = \int_{-\infty}^{+\infty} \psi^*(\mathbf{x})_{op} H_{op} \Psi(t, \mathbf{x}) d\mathbf{x} \exp \left[ \int_{-\infty}^{+\infty} \psi^*(\mathbf{x})_{op} \Psi(t, \mathbf{x}) d\mathbf{x} \right],$$

and

$$(24) \quad \mathcal{H}_{op} \chi[\Psi(t, \mathbf{x})] = \int_{-\infty}^{+\infty} \psi^*(\mathbf{x})_{op} H_{op} \Psi(t, \mathbf{x}) d\mathbf{x} \left( \exp \left[ \int_{-\infty}^{+\infty} \Psi(t, \mathbf{x}) \psi^*(\mathbf{x})_{op} d\mathbf{x} \right] \right) \chi_0,$$

as a consequence of (14), we have proven (22).  $H_{op}$  denotes the hamiltonian operator of the particle.

By the quantization of the quasi-irrotational motions of the Madelung fluid, we were able to get two kinds of states,  $\chi_{\Phi}$  and  $\chi[\Phi]$ , the first kind being closely related to the ordinary quantal description of one particle and the second kind being more closely associated with the unquantized fluid model. Indeed, in the quantized fluid theory  $\psi(\mathbf{x})_{op}$  is the operator that corresponds to the quantity  $\Psi(\mathbf{x})$  of the unquantized fluid, and the  $\psi(\mathbf{x})_{op}$  have definite values  $\Phi(\mathbf{x})$  in the state  $\chi[\Phi]$ .  $\chi_{\Phi}$  is associated to the creation operator  $a_{\Phi}^*$



defined by the equations (13) of section 3

$$(25) \quad \chi_{\Phi} = a_{\Phi}^* \chi_0, \quad \left( \int_{-\infty}^{+\infty} |\Phi|^2 dx = 1 \right).$$

We have obviously

$$(26) \quad \chi[\Phi] = \exp[(a_{\Phi}^* \chi_0)].$$

Therefore  $\chi_{\Phi}$  and  $\chi[\Phi]$  are special cases of a more general type of state  $\chi[\Phi, F]$

$$(27) \quad \chi[\Phi, F] = F(a_{\Phi}^*) \chi_0,$$

$$(28) \quad F(z) = \sum_{n=0}^{\infty} c_n z^n.$$

It is easily seen that, with a normalized  $\Phi$ , we have

$$(29) \quad \int |\chi[\Phi, F]|^2 d\omega = \sum_{n=0}^{\infty} |c_n|^2 n!.$$

$$(30) \quad \int \chi^*[\Phi, F] \mathcal{K}_{op} \chi[\Phi, F] d\omega = \langle K_{op} \rangle_{\Phi} \int |F'(a_{\Phi}^* \chi_0)|^2 d\omega = \\ = \langle K_{op} \rangle_{\Phi} \int |\chi[\Phi, F']|^2 d\omega = \langle K_{op} \rangle_{\Phi} \sum_{n=1}^{\infty} n |c_n|^2 n!.$$

$F'$  denoting the derivative of  $F$ . It follows from (30) that the expectation values of the  $\mathcal{K}_{op}$  in the fluid state  $\chi[\Phi, F]$  are proportional to the expectation values of the corresponding  $K_{op}$  in the particle state  $\Phi$ , the factor of proportionality being independent of  $K_{op}$ .

The operators  $\psi_{op}$  and  $\psi_{op}^*$  correspond to the variable  $\Psi$  and  $\Psi^*$  of the unquantized Madelung fluid, respectively. The procedure of quantization consists in the introduction of the commutation rules (5) or (9), that do not involve the Planck constant. *In other words, the quantization consists in the replacement of the c-number, variables of the theory of the Madelung fluid by q-numbers with non commutative multiplication, the rules of commutation being independent of the quantum of action. The replacement of the c-numbers by the q-numbers introduces the turbulence.*

We have already shown that the so-called second quantization methods are not essentially of a quantal nature <sup>(6)</sup>. They can be applied to the classical mechanics <sup>(7)</sup> and to the classical field theories <sup>(9)</sup>. The quantization of the

(9) M. SCHÖNBERG: *Nuovo Cimento*, **10**, 1597 (1953).

motions of the Madelung fluid may also be regarded as the application of the second quantization methods in order to get a stochastical generalization that includes turbulence.

### RIASSUNTO (\*)

Si discute un nuovo modello idrodinamico per l'equazione di Schrödinger. Il nuovo modello differisce da quello di Madelung per l'esistenza della turbolenza. Esso segue direttamente dall'interpretazione ordinaria della meccanica quantistica, con l'introduzione di operatori per le densità di carica e di corrente e le componenti del tensore degli sforzi nel formalismo per una singola particella. Si sviluppa il modello per qualsiasi valore dello spin. Il fluido di Madelung corrisponde al moto medio dello speciale mezzo turbolento. Il potenziale quantico appare come combinazione di una pressione con termini derivanti dalla turbolenza. Si mostra che la quantizzazione del moto del fluido di Madelung introduce il giusto tipo di turbolenza. Le traiettorie della teoria di de Broglie-Bohm appaiono come traiettorie del moto medio del mezzo turbolento.

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## Sulle temperature nucleari.

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(ricevuto il 21 Maggio 1954)

**Riassunto.** — Si esaminano alcune esperienze di evaporazione nucleare e si mostra che interpretandoli con la formula di Weisskopf i risultati sono in sensibile accordo.

Diverse esperienze sono state fatte negli ultimi anni per determinare le temperature dei nuclei eccitati, con risultati spesso in notevole disaccordo tra di loro. I risultati più discordanti sono stati trovati nelle esperienze indirette, in cui vengono esaminati gli spettri di emissione nelle reazioni  $A(x, n)B$ , ed interpretati mediante la teoria evaporativa. I proiettili  $x$  usati, di vario tipo, hanno generalmente energie di pochi MeV, per cui tutti gli autori hanno supposto valida l'ipotesi di Bohr, trascurando completamente una possibile emissione diretta. Ma se si tiene conto del principio di Pauli che, nel caso appunto di energie di incidenza molto basse, proibisce la maggior parte degli urti, si trova che il libero cammino medio in materia nucleare è anche a queste energie confrontabile con le dimensioni nucleari; dal punto di vista sperimentale si osserva che nella parte finale gli spettri sono troppo alti perchè tutti i neutroni possano essere stati emessi per evaporazione; si può pensare che i neutroni in eccesso siano stati emessi direttamente. Sorge perciò il dubbio che anche nella parte iniziale dello spettro una notevole emissione diretta, tale da alterare sensibilmente il fenomeno, possa precedere l'evaporazione; ciò rende necessario descrivere inizialmente la reazione come un processo di knock-on, come è stato fatto ad energie maggiori. Naturalmente tale modo di procedere potrà dare qui soltanto l'ordine di grandezza ed il limite superiore dell'emissione diretta.

I calcoli sono stati condotti con il metodo Montecarlo, con le formule usate

da DE SABBATA, MANARESI e PUPPI <sup>(1)</sup>, ma proiettando il nucleo su di un piano come BERNARDINI, BOOTH e LINDENBAUM <sup>(2)</sup>; è stato fatto il caso standard di neutroni di 15 MeV contro nuclei  $A=100$ . Dopo trenta casi la percentuale dei neutroni di knock-on per energia di emissione compresa tra 0 e 6 MeV è circa del 10%; ma usando per la parete del nucleo il coefficiente di riflessione che si ottiene adottando integralmente il modello ottico, come è più probabile a queste energie (vedi <sup>(1)</sup>), tale percentuale si riduce a circa il 3% (dopo venti casi le percentuali erano praticamente identiche e ciò rende accettabili questi valori come ordine di grandezza). Si vede perciò che a causa del potere riflettente della parete nucleare l'emissione diretta può essere trascurata, almeno nella parte iniziale dello spettro, che è quella che interessa per il calcolo delle temperature nucleari (nei nuclei più leggeri le percentuali non dovrebbero aumentare sostanzialmente, perchè il coefficiente di riflessione per i neutroni rimane praticamente invariato).

Nelle esperienze più recenti si è trovato che lo spettro di emissione  $n(\varepsilon)$ , almeno per  $\varepsilon$  abbastanza piccole, può essere messo sotto la forma

$$n(\varepsilon) = \text{cost } \varepsilon \cdot \exp [-\varepsilon/T],$$

con  $T$  costante. In queste esperienze l'energia della particella incidente è sempre abbastanza grande perchè una notevole percentuale dei nuclei bombardati emetta due neutroni; lo spettro osservato è perciò la somma di due spettri parziali  $W'(\varepsilon)$  e  $W''(\varepsilon)$ . Se si suppone che i singoli spettri  $W'(\varepsilon)$  e  $W''(\varepsilon)$  abbiano forma maxwelliana si trova, con certe approssimazioni, che anche la loro somma è maxwelliana;  $T$  è allora approssimativamente la temperatura nella prima emissione, e deve soltanto essere corretta a causa della seconda emissione. A parte la poca credibilità di evaporazioni con spettro maxwelliano, specialmente ad energie così basse, i vari tipi di esperienze darebbero in questo modo temperature molto diverse per lo stesso nucleo. È perciò utile tentare di spiegare la forma approssimativamente maxwelliana di  $n(\varepsilon)$  con un diverso andamento delle singole  $W(\varepsilon)$ . Ammettiamo valida la formula di Weisskopf:

$$W(\varepsilon)d\varepsilon = \text{cost } \sigma(\varepsilon) \cdot \varepsilon \exp [2\sqrt{a}\sqrt{\varepsilon_{\max} - \varepsilon}]d\varepsilon.$$

Ad energie di pochi MeV la sezione d'urto  $\sigma(\varepsilon)$  non può essere supposta costante neppure per i neutroni; qui sono state usate formule tratte dal lavoro citato <sup>(1)</sup>. Con calcoli numerici si trova che, per  $\varepsilon_{\max} \sim 15$  MeV, circa l'80% dei nuclei emette due neutroni. Calcolata l'energia di emissione media nella prima evaporazione si può definire un valore medio per l'energia di emissione massima

(1) V. DE SABBATA, E. MANARESI e G. PUPPI: *Nuovo Cimento*, **10**, 1704 (1953).

(2) G. BERNARDINI, E. T. BOOTH e S. J. LINDENBAUM: *Phys. Rev.*, **88**, 1017 (1952).

nella seconda evaporazione  $\varepsilon''_{\max}$ , e costruire quindi  $W''(\varepsilon)$ ; normalizzata  $W''(\varepsilon)$  in modo che sia

$$\int_0^{\varepsilon''_{\max}} W''(\varepsilon) d\varepsilon = 0,8 \int_0^{\varepsilon'_{\max}} W'(\varepsilon) d\varepsilon,$$

si costruisce  $n(\varepsilon)$ . In fig. 1 è riportato un esempio: si vede che  $n(\varepsilon)$  è del tipo richiesto sperimentalmente;  $W''(\varepsilon)$  non può essere considerato una correzione a  $W'(\varepsilon)$ , costituendo invece la parte predominante di  $n(\varepsilon)$  nella zona in cui

si compiono i calcoli. Se si costruisce per ogni  $\varepsilon'_{\max}$  una curva di  $T$ , che ora è una costante empirica priva di significato fisico, in funzione di  $a$ , è possibile ricavare dalle  $T$  sperimentali le corrispondenti  $a$ , e quindi le temperature nucleari  $\vartheta$ , legate alle  $a$  e all'energia di eccitazione  $E$  dalla formula:  $a\vartheta^2 = E$ . Procedendo in questo modo si trova la seguente relazione empirica tra  $a$  e  $T$ :

$$a(T - T^*)^2 = \varepsilon''_{\max}.$$

$T^*$  è funzione di  $a$  e di  $\varepsilon'_{\max}$ , ma nei casi che interessano ( $\varepsilon'_{\max} < 20$  MeV e  $2 \text{ MeV}^{-1} < a < 16 \text{ MeV}^{-1}$ ) è sempre compreso tra 0,1 e 0,2 MeV. Il calcolo è affetto da un grande errore a causa della scarsa conoscenza della energia di legame nella se-

conda emissione, essendo il nucleo finale quasi sempre sconosciuto; tale errore è maggiore nei nuclei più leggeri. Sarebbe perciò inutile tener conto delle variazioni di  $T^*$  con  $a$  e  $\varepsilon'_{\max}$  ed è sufficiente prendere un valore medio  $T^* = 0,15$  MeV.

Dato che, come si è detto, la seconda emissione costituisce la parte fondamentale dello spettro, potrebbe non essere sufficientemente esatto costruire  $W''(\varepsilon)$  mediante una  $\varepsilon''_{\max}$  media: perciò in due casi, per  $a = 6 \text{ MeV}^{-1}$  e  $12 \text{ MeV}^{-1}$ , è stata fatta una verifica, consistente praticamente in una evaporazione di tipo Montecarlo. Le differenze risultano di qualche per cento, cioè nettamente dentro gli errori di misura; viene comunque accentuato l'andamento rettilineo di  $\ln [n(\varepsilon)/\varepsilon]$ .

Nel caso dei nuclei più leggeri occorre tener conto della emissione, nella prima evaporazione, di particelle diverse da neutroni; questo si fa risentire particolarmente se le particelle incidenti sono protoni. Esaminiamo i risultati dei vari autori.

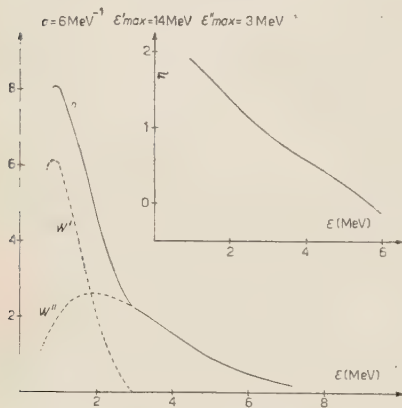


Fig. 1. - Esempio di costruzione di  $n(\varepsilon)$ : è riportato anche l'andamento di  $\eta = \ln [n(\varepsilon)/\varepsilon]$  (le ordinate sono in unità arbitrarie).



GUGELOT <sup>(3)</sup>: usa protoni di 16 MeV. Nel caso dell'alluminio e del ferro la prima emissione è costituita praticamente solo da protoni; infatti la grande energia di reazione negativa del processo (p, n) ( $\sim -5,5$  MeV in entrambi i casi) lo rende estremamente improbabile; neutroni si devono avere solo nella seconda emissione. In questo caso è  $T^* = 0$ . Nelle reazioni con nuclei più pesanti l'emissione di protoni è invece trascurabile. Complessivamente si ottiene:

Nucleo finale	<sup>26</sup> Al	<sup>55</sup> Fe	<sup>102</sup> Pd	<sup>196</sup> Hg	<sup>204</sup> Pb
$T$ (MeV) . . . . .	1,3	0,95	0,77	0,77	0,70
$T - T^*$ (MeV) . . . . .	1,3	0,95	0,62	0,62	0,55
$\varepsilon''_{\max}$ (MeV) . . . . .	2	2,5	4,0	4,1	5,2
$a$ { valore GUGELOT	3,2	6,5	12,5	14,5	15
(MeV <sup>-1</sup> ) { valore corretto .	$1,5 \pm 0,6$	$2,5 \pm 1$	$10 \pm 4$	$11 \pm 4$	$17 \pm 7$

GRAVES e ROSEN <sup>(4)</sup>: usano neutroni di 14 MeV. Solo nel caso dell'alluminio si ha una sensibile emissione di protoni: nel calcolo di  $T^*$  occorre tener conto di questo fatto. Complessivamente si ha:

Nucleo finale	<sup>26</sup> Al	<sup>55</sup> Fe	<sup>63</sup> Cu	<sup>65</sup> Zn	<sup>107</sup> Ag	<sup>111</sup> Cd	<sup>117</sup> Sn	<sup>196</sup> Au
$T - T^*$ (MeV) . . . . .	0,85	0,61	0,52	0,58	0,48	0,51	0,41	0,51
$\varepsilon''_{\max}$ (MeV) . . . . .	1,7	2,5	3	2,8	2,8	2,8	2,5	3,6
$a$ { valori G.R. . .	16	25	24	26	36	32	46	32
(MeV <sup>-1</sup> ) { valori corretti .	2,3	7	8	8	12	10,5	15	14
	$\pm 1$	$\pm 3$	$\pm 3$	$\pm 3$	$\pm 3,5$	$\pm 3$	$\pm 4,5$	$\pm 4$

GROSSKREUTZ <sup>(5)</sup>: usa deutoni di 10 MeV nella reazione:



L'emissione di protoni è trascurabile; è qui  $\varepsilon'_{\max} = 17$  MeV. Risulta

$$\varepsilon''_{\max} = 4 \text{ MeV}; \quad T - T^* = 0,85 \text{ MeV}; \quad a = 5 \pm 2 \text{ MeV}^{-1}.$$

FELD <sup>(6)</sup>: interpretando esperienze con neutroni di 3 e 1,5 MeV trova per il tungsteno il valore  $a \sim 12 \text{ MeV}^{-1}$ .

<sup>(3)</sup> P. GUGELOT: *Phys. Rev.*, **81**, 51 (1950).

<sup>(4)</sup> R. GRAVES e E. ROSEN: *Phys. Rev.*, **89**, 343 (1953).

<sup>(5)</sup> J. C. GROSSKREUTZ, *Phys. Rev.*, **76**, 482 (1949).

<sup>(6)</sup> B. T. FELD: *Phys. Rev.*, **75**, 1115 (1949).

LEVINTAL, MARTINELLI e SILVERMAN<sup>(7)</sup>: emissione di protoni dall'alluminio bombardato con protoni di 30 MeV. Trovano accordo qualitativo usando la formula di Weisskopf con  $a = 2,2 \text{ MeV}^{-1}$ ; si ha però una troppo grande emissione di protoni sia all'inizio che alla fine dello spettro. Questo è comprensibile, se si pensa che in questo caso l'emissione per knock-on deve essere notevole: ciò fa sì che si abbiano troppi protoni ad alta energia, mentre d'altra parte l'energia di eccitazione che va in evaporazione diminuisce, e questo sposta a sinistra il massimo della curva di evaporazione.

Nell'esaminare le esperienze di GRAVES e ROSEN non sono stati presi in esame i risultati nel caso del piombo e del bismuto; le  $a$  calcolate da GRAVES e ROSEN erano molto piccole e questo faceva loro pensare che la curva di  $a$  in funzione di  $A$  avesse un massimo, e che diminuisse fortemente per grandi  $A$ . Sviluppando i calcoli come sopra, la diminuzione si manterrebbe anche dopo la correzione, ottenendosi per  $a$  rispettivamente i valori 10,5 e 6  $\text{MeV}^{-1}$  (una diminuzione continua sarebbe però in contrasto con il valore  $a = 12 \text{ MeV}^{-1}$  dato da WEISSKOPF per  $A = 231$ ). Si può pensare che tali valori non indichino una diminuzione continua di  $a$ , ma siano valori anomali dovuti al fatto che il  $^{209}\text{Bi}$  e l'isotopo 208 del piombo sono numeri magici per i neutroni ( $N=126$ ). HUGHES, GARTH e LEVIN<sup>(8)</sup> con misure dirette hanno trovato valori della densità dei livelli  $\omega$  notevolmente più bassi della media per nuclei che hanno un numero di neutroni uguale ad un numero magico più uno; i risultati di GRAVEN e ROSEN potrebbero indicare una analoga diminuzione di  $a$ , e quindi di  $\omega$ , nel caso che il numero dei neutroni sia uguale ad un numero magico meno uno. Infatti il nucleo finale, ad esempio, nell'evaporazione del bismuto è il  $^{208}\text{Bi}$ , con  $N=125$ ; e dato che il valore di  $a$  nella prima evaporazione ha pochissima influenza sul risultato finale, i dati sperimentali si interpretano bene con un valore di  $a$  normale per il  $^{209}\text{Bi}$ , cioè nella prima evaporazione  $a \sim 5 \text{ MeV}^{-1}$  per il  $^{208}\text{Bi}$ , cioè nella seconda evaporazione. Un ragionamento analogo si può fare nel caso di  $^{208}\text{Pb}$  e  $^{207}\text{Pb}$ ; ma in questo caso il  $^{208}\text{Pb}$  è contenuto nel piombo naturale solo per il 52%; è perciò spiegabile che, entrando nella seconda evaporazione anche nuclei normali, la  $a$  misurata risulti intermedia tra il valore normale e quello del  $^{208}\text{Bi}$ .

## Conclusioni.

Le esperienze esaminate, interpretate con formule maxwelliane, danno come risultato valori di  $a = E/\beta^2$  fortemente dispersi (fig. 2a); usando la formula di Weisskopf le  $a$  risultano invece approssimativamente disposte lungo

(7) C. LEVINTAL, E. A. MARTINELLI e A. SILVERMAN: *Phys. Rev.*, **78**, 199 (1950).

(8) D. J. HUGHES, R. C. GARTH e J. G. LEVIN: *Phys. Rev.*, **91**, 1423 (1953).

una curva continua (fig. 2b). Due valori troppo bassi di  $a$  che si ottengono dalle esperienze di GRAVES e ROSEN, possono essere dovuti alla presenza nel-



Fig. 2a. — Valori di  $a$  dati dai vari autori.

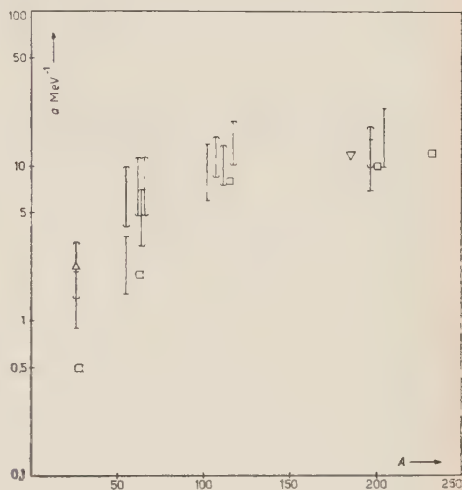


Fig. 2b. — Valori di  $a$  corretti.

l'evaporazione di nuclei con un numero di neutroni uguale ad un numero magico meno uno.

L'emissione diretta, almeno per energie fino a 15 MeV, non modifica la parte iniziale dello spettro; probabilmente è invece sensibile nella parte finale.

Ringrazio sentitamente il prof. G. PUPPI e il dott. DE SABBATA che mi hanno consigliato e incoraggiato a compiere questo lavoro.

#### SUMMARY (\*)

Some experiences of nuclear evaporation are discussed and it is shown that interpreting them with Weisskopf's formula the results agree fairly well.

(\*) Traduzione a cura della Redazione.

## LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

### A proposito delle esperienze di interazione ionosferica delle radioonde.

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(ricevuto il 16 Aprile 1954)

Ho avuto occasione recentemente di analizzare i lavori del dott. CUTOLO sull'interazione, pubblicati a più riprese dal *Nuovo Cimento*, ed in particolare di soffermarmi, negli ultimi due (1° Agosto 1952 e 1° Luglio 1953), sul fenomeno detto di « autointerazione ». Poichè il soggetto, che senza dubbio è interessante, presenta ancora molti punti di incertezza e merita ulteriore attenta indagine, potrà forse riuscire utile che io chiarisca qui il mio pensiero in proposito, tanto più che esso non si accorda senza riserve con le opinioni manifestate dal dott. CUTOLO in quei lavori e l'espressione di un diverso punto di vista ha in ogni caso un contributo di utilità.

Eccessivamente ottimistica sembra in sintesi la valutazione di poter dedurre dai risultati di quelle esperienze il valore del campo magnetico terrestre nella l'alta atmosfera, con tanta approssimazione: l'uno per mille è infatti dichiarato nella nota del 1° Luglio 1953 e vi è giudicato possibile giungere addirittura all'uno su diecimila. Se si esaminano invece i risultati diretti delle esperienze, già per quelle di girointerazione, che sono le più sicure, l'approssimazione raggiungibile sembra assai minore. Anche per le esperienze del 13 Luglio 1949, le quali sono

quelle che con maggiore evidenza hanno fornito i due caratteristici massimi (fig. 2e della nota del 1° Maggio 1952, riportata con omissione di alcuni punti come fig. 1 della nota del 1° Luglio 1953), data la dispersione dei punti sperimentali sembra già ottimistico pensare che la posizione dei massimi possa venir localizzata con un'approssimazione del 0,5%. Più dubbio è l'apprezzamento delle esperienze del 3 e 6 Luglio 1949 (fig. 2b e 2c della nota del 1° Maggio 1952), per le quali l'incertezza di localizzazione dei massimi non può essere ragionatamente valutata minore dell'1 o 2%. Poichè la relazione che lega il valore di  $H$  a quello della frequenza giromagnetica  $f_r$  (assunta pure come valore medio tra quelli corrispondenti ai due massimi) è una relazione di semplice proporzionalità, identica è l'incertezza prevedibile nella determinazione di  $H$ . Perciò nei valori dati per  $H$  nella tabella III della nota del 1° Luglio 1953 già la terza cifra decimale non può essere riguardata come significativa; rientrano dunque nell'incertezza le differenze tra il valore calcolato con la formula di Schmidt in base a quello alla superficie terrestre ed i valori dedotti dalle esperienze di girointerazione; cosicchè queste non sembra rie-



scano per ora a portare un maggiore affinamento nella determinazione del valore di  $H$  ad alta quota.

Analoghe considerazioni possono farsi sulle esperienze di autointerazione, le quali in più lasciano adito a vari interrogativi. Posto che fosse esattamente nota per teoria la forma della curva con la quale devono essere interpretati i punti sperimentali (fig. 2 della nota del 1° Luglio 1953), già parrebbe incerto anche qui sperare in un'approssimazione migliore dell'1% nella localizzazione dei massimi. Ma è appunto la forma di questa curva che lascia i maggiori dubbi. Perché dev'essere essa foggata ad M? Può essere data una dimostrazione teorica a sostegno di tale assunzione? Che cosa l'Autore pensa debba accadere al di fuori del campo di frequenze considerato nella figura? Dovrà la profondità di modulazione del segnale ricevuto continuare a decrescere sempre più, via via che ci si allontana dalla girofrequenza?

Questi interrogativi lasciano per ora seri dubbi sulla certezza di poter interpretare i risultati delle esperienze come dovuti ad un fenomeno di autogirointerazione. Sarebbe pertanto auspicabile che una teoria *fisica* del fenomeno, non una semplice trattazione modellistica basata sul presupposto generico della non linearità del mezzo, venisse presto a portar luce sull'argomento. Sembra purtroppo che le difficoltà per sviluppare una simile teoria siano molto gravi.

Non minore perplessità provo del resto di fronte ad altri aspetti del complesso di osservazioni nelle quali è stata ravvisata la manifestazione di un fenomeno di autointerazione. In ispecie mi fa meditare l'apparente rapido aumento dell'entità del fenomeno al crescere della frequenza di modulazione (fig. 2 della nota del 1° Agosto 1952); se infatti è vera, come sembra doversi ammettere, l'affinità del fenomeno di autointerazione con quello di interazione, dovrebbe avvenire esattamente il contrario: cioè la diminuzione della profondità di mo-

dulazione prodotta dall'autointerazione sull'onda che si propaga dovrebbe essere massima con basse frequenze di modulazione e decrescere rapidamente col crescere della frequenza, fino a scomparire già per un migliaio di hertz. Si può notare che l'Autore non fa alcun cenno, nel riferire di quelle prime esperienze, all'errore sistematico crescente con la frequenza, che è provocato dalla selettività del ricevitore. Esso non dovrebbe essere trascurabile con un ricevitore così selettivo qual'è il BC314. Di quale entità era esso realmente, alle varie frequenze di modulazione?

Non intendo con tutti questi interrogativi porre dubbi sulla possibilità di esistenza del fenomeno di autointerazione. Non si può aver dubbi in proposito, quando si pensi alle cause fisiche del corrispondente ben noto fenomeno dell'interazione tra campi elettromagnetici di frequenze diverse. È probabile anche che, in favorevoli circostanze, il fenomeno di autointerazione possa presentare determinate caratteristiche di selettività nell'intorno della girofrequenza. Ma gli effetti di variazione della profondità di modulazione prodotti dall'autointerazione sono generalmente mescolati a quelli prodotti dal ben noto fenomeno di « fluttuazione selettiva », dovuto all'interferenza di più componenti prodotte da radiazioni che hanno seguito nel propagarsi percorsi di lunghezza diversa. Risulta da quest'ultimo fenomeno un'alterazione nei rapporti di ampiezza e di fase delle diverse righe dello spettro di un'oscillazione modulata, l'una o l'altra delle quali viene più o meno esaltata o ridotta. Si ha così una deformazione dello spettro che segue una legge periodica in funzione della frequenza, più o meno complessa e diversa secondo il numero di componenti che interferiscono e le loro differenze di cammino, e nascono di conseguenza variazioni della profondità di modulazione e deformazioni spesso rilevanti del segnale modulante. Questi effetti di fluttuazione selettiva sono frequentissimi nella rice-



zione delle onde corte e sono stati riscontrati e studiati recentemente anche su onde centimetriche <sup>(1)</sup>. In quest'ultimo caso non vi è dubbio che debba trattarsi di interferenza tra « raggi » diversi e non di variazioni con la frequenza e nel tempo delle caratteristiche di propagazione del mezzo.

Ora accade che, mentre è da presumere, per il parallelo col fenomeno di interazione, che l'autointerazione non debba produrre diminuzioni della profondità di modulazione maggiori di un 10%, il fenomeno di fluttuazione selettiva determina variazioni assai più rilevanti della profondità; nel caso di una frequenza pura modulante si può giungere da un lato all'annullamento della modulazione nel segnale ricevuto (scomparsa delle due righe laterali) o dall'altro a valori di profondità di modulazione maggiori del 100% (ampiezza della riga centrale maggiore della somma delle due laterali). Questa differenza quantitativa tra i due effetti è serio intralcio nello studio sperimentale dell'autointerazione.

Si potrebbe pensare a separarli con un procedimento statistico, dato che le variazioni di profondità di modulazione dovute alla fluttuazione selettiva sono ora in più, ora in meno, ed invece quelle dovute all'autointerazione dovrebbero essere in meno, non sempre, ma presumibilmente almeno in casi bene scelti. Ma non è certo anzitutto che le prime mediamente si annullino nel tempo, quando sono molto grandi, e d'altra parte è dub-

bio se sia possibile mettere in evidenza una variazione sistematica di qualche unità per cento nella profondità di modulazione, imputabile all'autointerazione, con uno scarto quadratico medio dei valori sperimentali notevolmente maggiore. La cosa sarebbe più facile se si riuscisse a dimostrare teoricamente, in modo certo, che l'effetto di autointerazione è massimo per le frequenze di modulazione più basse, perchè per queste l'effetto della fluttuazione selettiva è minimo.

Altra via da tentare per cercare di mettere in evidenza l'effetto di autointerazione è quella di variare ciclicamente e abbastanza rapidamente la potenza emessa, in successivi salti con rapporto 1:2 per ciascuno, mantenendo costanti le frequenze della modulazione e della portante. Sull'effetto di fluttuazione selettiva non dovrebbero avere alcuna influenza le variazioni di potenza (purchè naturalmente il livello del segnale alla ricezione sia sempre convenientemente al disopra del livello dei disturbi), mentre ne dovrebbero avere ovviamente su quello di autointerazione. I salti di potenza dovrebbero avvenire con ritmo di almeno uno al minuto, in modo che si possa supporre poco variare le condizioni della ionosfera nel giro di un certo numero di cicli di variazione della potenza.

È possibile che per questa o per altre vie, che forse si possono ancora escogitare, ulteriori esperienze riescano a dare qualche maggior ragguaglio sul fenomeno di autointerazione. Comunque in questo campo difficile di ricerche una grande cautela nell'interpretare ed accettare i risultati delle esperienze non sarà mai superflua.

<sup>(1)</sup> A. B. CRAWFORD e W. C. JAKES, Jr.: *Bell S. T. J.*, 31, 68 (1952).

[Il Dott. M. CUTOLO, cui avevamo comunicato alla fine di Aprile questa lettera, risponde alle osservazioni in essa contenute con scritto pervenutoci quando già il fascicolo era in macchina. *La Direzione.*]

## Assorbimento in risonanza di neutroni nel sodio.

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(ricevuto il 18 Maggio 1954)

Si assume generalmente che il sodio sia in pratica un puro « assorbitore  $1/v$  » per i neutroni, poichè la sua prima risonanza è situata a circa 3000 eV, e in tale regione d'energia il fenomeno predominante è lo scattering di risonanza, piuttosto che l'assorbimento. Perciò, misure di attivazione del sodio sono usate sia per determinare il numero totale di neutroni per unità di volume in un mezzo rallentante, sia per misurare gli integrali di risonanza di vari elementi mediante la determinazione del « cadmium ratio » <sup>(1)</sup> (ossia del rapporto tra l'attivazione totale e l'attivazione dovuta a neutroni di energia superiore a quella del « taglio » del cadmio).

Per verificare la validità della precedente ipotesi, il « cadmium ratio » relativo al sodio è stato confrontato con quello del boro, che è notoriamente un assorbitore  $1/v$ . Il fascio collimato di neutroni emergente dal centro della pila di Kjeller (JEEP) è stato usato sia allo scopo di irraggiare il sodio con e senza interposizione di uno schermo di cadmio, sia per misurare nelle stesse due condizioni il rapporto dei conteggi per un contatore proporzionale a  $\text{BF}_3$ . I risul-

tati trovati sono

$$\text{CdR}_B = 48,1 \pm 0,25$$

$$\text{CdR}_{\text{Na}} = 41,5 \pm 0,8.$$

La differenza tra i due rapporti  $\text{CdR}_B$  e  $\text{CdR}_{\text{Na}}$ ; trovati rispettivamente con boro e sodio, indica che deve esistere per il sodio una attivazione di risonanza facilmente misurabile.

Il valore dell'integrale di assorbimento in risonanza del sodio corrispondente alla precedente differenza tra  $\text{CdR}_B$  e  $\text{CdR}_{\text{Na}}$  è stato calcolato e risulta <sup>(2)</sup>

$$\Sigma_a = 0,037 \text{ barn}.$$

Per tale calcolo la sezione d'urto di risonanza  $\sigma_{\text{ris}}$  è stata scelta come la differenza tra la effettiva sezione d'urto di cattura del sodio e quella che si avrebbe in base alla pura legge  $1/v$ .

È da attendersi che tale piccolo valore dell'integrale di risonanza causi errori trascurabili quando si misura il numero totale di neutroni per unità di volume in mezzi dove oltre ai neutroni termici sono presenti anche neutroni di più alta energia. Però il « cadmium ratio » del sodio è apprezzabilmente differente da quello relativo al boro. Quindi il consi-

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<sup>(1)</sup> S. P. HARRIS, C. O. MUELHAUSE e G. E. THOMAS: *Phys. Rev.*, **79**, 11 (1950).

<sup>(2)</sup> D. POPOVIĆ: in corso di pubblicazione in *Zeits. f. Naturf.*

derare il « cadmium ratio » del sodio come caratteristico di un assorbitore  $1/v$  per determinare i valori di integrali di risonanza per altri elementi può causare sensibili errori, specialmente per quegli elementi che hanno piuttosto deboli assorbimenti in risonanza.

Usando per l'integrale dello scattering di risonanza del sodio il valore  $\Sigma_s = 51$

barn <sup>(1)</sup>, si ha

$$\frac{\Gamma_\gamma}{\Gamma} \simeq \frac{\Sigma_a}{\Sigma_a + \Sigma_s} \simeq 10^{-3}.$$

in accordo con il risultato precedentemente noto <sup>(3)</sup>  $\Gamma_\gamma/\Gamma < 10^{-2}$ .

<sup>(3)</sup> C. T. HIBDON, C. O. MUELHAUSE e W. WOOLF: *Phys. Rev.*, **77**, 730 (1950).

***P-Wave Pion Nucleon Scattering by Variational Method.***

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(ricevuto il 19 Maggio 1954)

We have calculated the  $P$  wave phase shifts in pion nucleon scattering using the extended source approximation and the variational method developed by CINI and FUBINI (<sup>1</sup>). The pseudoscalar interaction Hamiltonian is, in this approximation (<sup>2</sup>)

$$(1) \quad H = \sqrt{4\pi} \frac{f}{\mu} \int \varrho(x) \sigma \cdot \nabla \tau_i \varphi_i(x) d^3x$$

where  $\varrho(x)$  is the source density normalized to unity.

The first approximation of the variational method of CF gives for the reaction matrix  $K$  (<sup>3</sup>)

$$(2) \quad K = (K_2)^2 (K_2 - K_4)^{-1},$$

$K_2$  and  $K_4$  being respectively the second and fourth order perturbation approximations to  $K$ . (2) is in general a matrix equation. However, for pion nucleon scattering  $K$  can be chosen diagonal in  $E$ ,  $J$ ,  $L$  and  $T$  and consequently (2) reduces to an algebraic equation for the matrix elements.

Charge and mass renormalizations are performed on each perturbation term (<sup>4</sup>). Equation (2) then leads to the following expressions for the  $P$  wave phase shifts:

$$(3a) \quad \text{tg } \alpha_{33} = \frac{2y}{1 - 2A_- - 4A_+},$$

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(<sup>1</sup>) M. CINI and S. FUBINI: *Nuovo Cimento*, **11**, 142 (1954); referred to as CF.

(<sup>2</sup>) W. PAULI: *Meson Theory of Nuclear Forces* (New York, 1946), p. 34.

(<sup>3</sup>) B. A. LIPPMANN and J. SCHWINGER: *Phys. Rev.*, **79**, 569 (1950). The phase shifts follow directly from the matrix elements of  $K$  by the relation  $\text{tg } \alpha_{ij} = -(k_0 \omega_0 / 4\pi^2) K_{ij}$ .

(<sup>4</sup>) See footnote (<sup>17</sup>) of reference (<sup>1</sup>). A similar procedure is followed by CHEW in the Tamm-Dancoff calculation of reference (<sup>6</sup>). The question of renormalization of a non-perturbation theory will be discussed more fully in a forthcoming work.

$$(3b) \quad \operatorname{tg} \alpha_{31} = \operatorname{tg} \alpha_{13} = \frac{-y}{1 + \Delta_- - \Delta_+},$$

$$(3c) \quad \operatorname{tg} \alpha_{11} = \frac{-4y}{1 + 4\Delta_- + 2\Delta_+},$$

where

$$(4a) \quad y = \frac{2}{3} \left( \frac{f}{\mu} \right)^2 \frac{k_0^3}{\omega_0} v^2(k_0),$$

$$(4b) \quad \Delta_{\pm} = \frac{2}{3\pi} \left( \frac{f}{\mu} \right)^2 \int dk \frac{k^4}{\omega^3} \left( \frac{\omega_0}{\omega \pm \omega_0} \right) v^2(k),$$

$v(k)$  being the cut-off factor and  $\omega_0$  the energy of the incident meson. Expressions (4) appear in the fourth order perturbation result of BLAIR and CHEW (5). It is interesting to note that the equations (3) are identical with the results obtained by CHEW (6) from a variational solution of the Tamm-Dancoff equation, except for the terms in  $\Delta_{\pm}$ . The latter represent the effect of processes in which three mesons appear in an intermediate state, and for  $\alpha_{33}$  contribute about 70% as much as the terms in  $\Delta_{\pm}$ .

Using equations (3) with a square cut-off at the nucleon mass and a coupling constant  $f^2=0.065$  (corresponding to a value of the usual pseudoscalar coupling constant  $g^2=11.7$ ) we have been able to fit the  $P$  wave part of the GLICKSMAN set of phase shifts (7), as shown in Fig. 1. The Glicksman values for the important

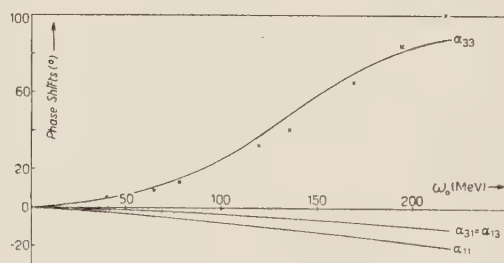


Fig. 1. —  $P$ -wave phase shifts as a function of incident meson energy. The solid lines represent our results. The crosses indicate Glicksman's solution and the solid squares the values quoted by BODANSKY *et al.*

phase shift  $\alpha_{33}$  at five energies are indicated by crosses on the diagram; the value  $100^\circ$  at 217 MeV is somewhat less significant than the others because it does not take account of the small  $D$  wave contribution indicated by Glicksman's data at that energy. The other three phase shifts  $\alpha_{31}$ ,  $\alpha_{13}$ ,  $\alpha_{11}$  are taken as zero in Glicksman's solution. The squares at lower energy on the diagram represent results cited by BODANSKY, SACHS and STEINBERGER (8); the other phase shifts are small

(5) T. S. BLAIR and G. F. CHEW: *Phys. Rev.*, **90**, 1065 (1953).

(6) G. F. CHEW: *Phys. Rev.*, **89**, 591 (1953).

(7) M. GLICKSMAN: *Phys. Rev.*, (to be published).

(8) D. BODANSKY, A. M. SACHS and J. STEINBERGER: *Phys. Rev.*, **93**, 1367 (1954).



but not zero in these data. It must be emphasized that the Glicksman solution is only one of the possible sets of phase shifts compatible with the experimental data presently available, and is itself subject to variation due to uncertainties in the data <sup>(9)</sup>.

Our results are very similar to those obtained by CHEW <sup>(10)</sup> using the same extended source theory and solving numerically an integral scattering equation. The values of the cut-off energy and coupling constant used by CHEW in fitting the Glicksman phase shifts are  $\omega_{\max}=5,6 \mu$ ,  $f^2=0.058$ . The similarity of the results seems to lend some further support to the validity of the extended source theory as applied to this problem. On the other hand, it may be noted that the first approximation of the variational method of CF, which is easily evaluated, gives results as good as those obtained by more complicated calculations.

Calculations extending the theory to include *S*-wave scattering are in progress.

We wish to express our gratitude to Prof. M. CINI for useful advice and many stimulating discussions.

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<sup>(9)</sup> For a discussion of the uncertainties in the phase shift analysis, see the *Proceedings of the 1953 Rochester Conference on High Energy Physics*.

<sup>(10)</sup> G. F. CHEW: *Phys. Rev.* (to be published). We are thankful for the opportunity of seeing Prof. CHEW's paper before publication.

## A Covariant Non-Adiabatic Equation for Nucleon-Pion Scattering.

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(ricevuto il 21 Maggio 1954)

The second order nucleon-pion Bethe-Salpeter equation contains implicit divergencies. By this we mean that the Neumann-Liouville iteration solution would contain divergent integrals. In a letter with the above title, FUBINI <sup>(1)</sup> has proposed a prescription for removing the implicit divergencies. Fubini's method is not consistent with the prescriptions of DYSON and SALAM for renormalising the  $S$ -matrix. The purpose of this letter is to show that the iteration solution can be used as a guide of how to do the subtractions in accordance with  $S$ -matrix theory. Of course, the iteration series will not converge in important cases; and so it has only a formal significance.

We use FUBINI's notation. For simplicity, we drop his graph I. That is, we replace  $S'_F$  defined by FUBINI's equation (4) by  $S_F$ . The formal treatment of  $S'_F$  is quite clear, but FELDMAN <sup>(2)</sup> has shown that severe difficulties arise in practice.

Define a 2nth order vertex part,  $A_n^{(1)}(p, k)$ , inductively by

$$(1) \quad \left\{ \begin{array}{l} A_0^{(1)} = \gamma^5, \\ A_{n+1}^{(1)}(p, k) = g^2 \int dk' \gamma^5 S_F(p - k') \gamma^5 S_F(p') A_n^{(1)}(p', k') A_F(k') \delta(p' + k' - p - k) dp'. \end{array} \right.$$

Then FUBINI's  $A^{(1)}$  is given formally by

$$(2) \quad A^{(1)}(p, k) = \sum_{n=0}^{\infty} A_n^{(1)}(p, k).$$

According to DYSON's <sup>(3)</sup> prescription, the convergent parts of the  $A_n^{(1)}$  satisfy

$$(3) \quad A_{n+1,c}^{(1)}(p, k) = g^2 \int dk' \gamma^5 S_F(p - k') \gamma^5 S_F(p') A_{n,c}^{(1)}(p', k') A_F(k') \cdot \delta(p' + k' - p - k) dp' - \gamma^5 L_n,$$

<sup>(1)</sup> S. FUBINI: *Nuovo Cimento*, **10**, 851 (1953).

<sup>(2)</sup> G. FELDMAN: *Proc. Roy. Soc.*, A **223**, 112 (1954).

<sup>(3)</sup> F. J. DYSON: *Phys. Rev.*, **75**, 1736 (1949).

where

$$(4) \quad \gamma^5 L_n = g^2 \int dk' \gamma^5 S_F(p_0 - k') \gamma^5 S_F(p') A_{c,n}^{(1)}(p', k') A_F(k') \delta(p' + k' - p_0) dp',$$

and  $i\gamma \cdot p_0 + M = 0$ ; so that  $L_n$  is an infinite constant. From (2) and (3), we have

$$(5) \quad A_c^{(1)}(p, k) = g^2 \int dk' \gamma^5 S_F(p - k') \gamma^5 S_F(p') A_F(k') \delta(p' + k' - p - k) \cdot A_c^{(1)}(p', k') dp' + \gamma^5(1 - L),$$

where  $L = \sum_1^\infty L_n$ .

We propose (5) as the integral equation to replace FUBINI's equations (7), (9) and (10).  $A_c^{(2)}$  is defined analogously. For (5) to have any sense, we must find a solution satisfying  $A_c^{(1)}(p_0, 0) = \gamma^5$ , for some finite  $L$ . For a similar equation, EDWARDS<sup>(4)</sup> has been able to find a solution with  $L=1$ . From equation (6), we see that such a solution would probably be unacceptable in our case.

In FUBINI's self-energy part  $\sum_{II}(p)$ , there are overlapping divergencies. These can be accounted for by defining instead

$$(6) \quad \sum_{II}(p) = (1 - L)g^2 \int \gamma^5 S_F(p - k) A_F(k) A_c^{(1)}(p - k, k) dk.$$

Equation (6) is justified by arguments similar to those used by SALAM<sup>(5)</sup> for the «  $b$ -divergencies » in the general case.  $\sum_{II}(p)$  gives FUBINI's  $\sum_{II,c}(p)$  after the usual subtraction for self-energy parts, and then one may apply the rest of FUBINI's equations.

Our conclusion, then, is that there are effectively no vertex part subtractions to be made. Our prescription no more guarantees convergent results than FUBINI's does; but, if these are obtained, our subtractions are justified formally by the usual renormalisation theory.

The author is grateful to Dr. ABDUS SALAM for discussions on the subject of this letter.

<sup>(4)</sup> S. F. EDWARDS: *Phys. Rev.*, **90**, 284 (1953).

<sup>(5)</sup> A. SALAM: *Phys. Rev.*, **82**, 217 (1951).

## Light Emission by Germinating Plants.

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(ricevuto il 29 Maggio 1954)

1. — It is known that the use of the photomultipliers as counters of pulses given by individual photons permits the attainment of the highest sensitivity for the detection of weak light pulses.

Some workers conducted extensive studies on the characteristics of RCA.931 photomultipliers or an equivalent thereof, and have investigated their possibilities as counters of single photons <sup>(1)</sup>.

Because of the small size of the photocathode, these photomultipliers are not very useful when it is desired to measure the light emitted by large surfaces as in the case of very weak emission per unit area.

In these cases photomultipliers EMI 6260 or RCA 5819 are more convenient, because of the large photocathode placed on the top of the tube, allowing a larger solid angle to be used in collecting light.

A paper on the behaviour of these photomultipliers as single photon counters has been previously published <sup>(2)</sup>.

By means of this very sensitive appa-

ratus we tried to detect a probable light emission from living vegetable tissues.

The research has been successful in

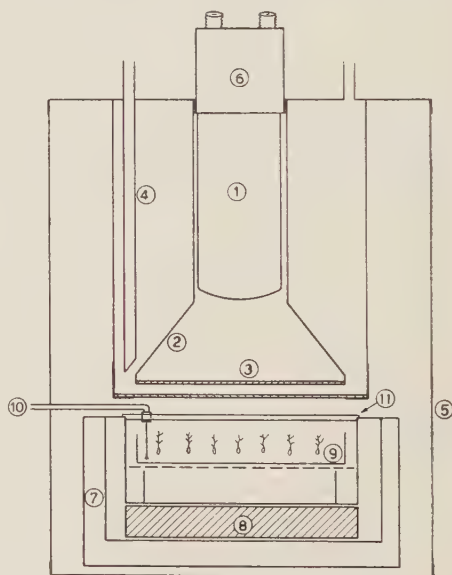


Fig 1. — Scheme of the photomultiplier-thermostate setting. A similar setting is used, cooled with dry ice. 1) Photomultiplier, 2) diffusing light guide, whitened with magnesium oxide, 3) glass, 4) water cooler, 5) light tight box, 6) socket containing the voltage divider, 7) thermostate box, 8) electric heating element, 9) glass plate containing the plants under study, 10) thermocouple thermometer 11) lucite lid.

<sup>(1)</sup> R. W. ENGSTROM: *J.O.S.A.*, **37**, 420 (1947); G. A. MORTON and J. A. MITCHELL: *RCA Rev.*, **9**, 632 (1948); R. WESTOO and T. WIEDLING: *Ark. för Fysik*, **1**, 269 (1949).

<sup>(2)</sup> L. COLLI, U. FACCHINI and A. ROSSI: *Nuovo Cimento*, **11**, 255 (1954).

the case of germinating seedlings of some common plants. Some of our preliminary results are given in this paper.

2. - The experimental equipment is shown in Fig. 1. In order to ensure a steady temperature, water circulation is arranged around the phototube. In some case the phototube was cooled with a ice-acetone freezing mixture, in order to reduce the thermoelectronic background.

The electronic equipment and the

plate, 14 cm in diameter, in the counting apparatus of Fig. 1. The plants averaged, when fresh, 60 g in weight.

Table I shows some results obtained in this way. The statistical errors are less than 5%.

The obtained results give good evidence for the existence of the luminescence.

The light intensity was found to depend on the temperature of the plants and it is not influenced by prior illumin-

TABLE I. - *Results on seedlings.*

Phototube	Kind of the plant used (6 days old)	Fresh weight	Temperature	Total pulses/min	Back-ground pulses/min	Effect pulses/min
N. 143, cooled with H <sub>2</sub> O	wheat	60 g	30° C	7 936	4 608	3 328
N. 195, cooled with dry ice	lentils	60 g	22° C	7 680	1 024	6 556
»        »	corn	60 g	22 °C	11 520	1 280	10 240
»        »	corn (grown in aseptic conditions)	60 g	22 °C	8 960	1 280	7 680

counting technique are the same as described in (2); pulses corresponding to single photoelectrons are recorded, above a given bias voltage and counted by a scaler scaling by 256.

We studied the seedlings of the following plants; wheat, corn, lentils, beans. The plants are grown in glass plates and are watered regularly. The growth of the seedlings took place in darkness, to avoid any excitation of phosphorescence. Under this conditions, practically no chlorophyll was formed and the well known red phosphorescence of the chlorophyll was not observed.

3. - In a first run of measurements about 300 seedlings were placed in a glass

ation of the plants. In the case of corn, measurements were made also with plants grown in aseptic conditions.

Seeds were sterilized for 20 minutes with a 0.3 percent HgCl<sub>2</sub> solution. It was established that the seeds were free of bacteria, as shown by the fact that no bacteria colonies appear when the seeds are grown on glucose peptone agar. Results obtained with these plants are like the ones obtained with the non sterilized plants. This indicates that the luminescence observed does not seem to be due to luminescent bacteria.

Some preliminary tests using colored filters showed that the most of the light observed corresponds to a wave length between 4 500  $\text{\AA}$  and 6 500  $\text{\AA}$ .



4. - A better accuracy was obtained by studying homogenized parts of the plants. In order to have a good reproducibility we were careful to standardize a method of grinding the different parts of the plants and to measure the luminescence of the pulp obtained. Ten grams of fresh material are used and ground for 8-12 minutes into a greenish opalescent pulp, which is heterogeneous because of the presence of solid parts of rather small dimensions.

This material is put in a glass plate, 14 cm in diameter and is diluted with water in the ratio of 5:1 in order to obtain a layer of a few mm of sufficiently transparent liquid.

It was noted that the addition of water does not alter the luminescence observed.

A good reproducibility was obtained with similar conditions.

From tests made with different quantities of pulp it may be deduced that the self absorption of light is not very big. It may be said that this effect diminishes the total amount of emitted light by a factor  $\sim 2$ .

Table II shows the results obtained for homogenized lentil stems, roots and seeds.

TABLE II. - *Luminescence intensity of different parts of lentil seedlings.*

Photomultiplier EMI 6260: Background 8000 pulses/minute. Statistical errors: less than 5%. Measurements taken on homogenized parts, 25 min after the manipulation. Age of the lentils: 6 days. Temperature: 20 °C.

	Fresh weight	Effect pulses/min
Stems . . . .	10 g	38400
Roots . . . .	10 g	116500
Seeds (without skin) . . .	10 g	38000

It is to be noted that the luminescence of the ground pulp decays in time.

Fig. 2 shows luminous intensity versus time in the case of homogenized stems. Similar results are obtained for roots and seeds.

In the case of homogenized pulps it is possible to obtain a rough estimate of the absolute intensity of the light emitted.

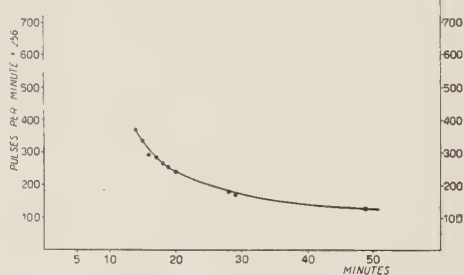


Fig. 2. - Luminescence decay versus time in the case of stem pulp. Fresh weight: 9 g plus 50 cm<sup>3</sup> of H<sub>2</sub>O. Age of the plants: 12 days. Temperature 20 °C. At time zero: the beginning of the grinding. Background of the photomultiplier 35 × 256 pulses/minute.

We studied a defined geometrical arrangement in order to have a calculated solid angle of about 10%. Assuming that the average yield of the photocathode is 6 electrons extracted for every incident photon, we obtain a light intensity corresponding to  $\sim 100\,000$  photons emitted per second and per gram of fresh weight at 20 °C.

5. - In the homogenized pulp a strong dependence of the emitted light intensity on the temperature was also observed: the light intensity versus the temperature increases till about 60-70 °C and then rapidly decreases.

This behaviour is characteristic of biochemical reactions when proteic systems are involved.

Finally we observed that the luminescence of the homogenized pulps increases many times with the addition of a small quantity of H<sub>2</sub>O<sub>2</sub>; for instance

a cubic centimeter of an  $\text{H}_2\text{O}_2$  solution in water having a concentration of about 1/30 000 added to 20 g of homogenized pulps raises the light intensity by a factor of 10 or more.

6. — Further work is planned in order to clarify the behaviour and the

nature of the observed bioluminescence.

Our thanks are due to Prof. G. BOLLA for useful advices, to Dr. C. POLVANI and to Drs. M. ORSENIGO and G. GUIDOTTI for continuous assistance. We finally thank Prof. E. N. HARVEY for his suggestions.

## Sui conteggi di granuli nelle emulsioni nucleari.

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(ricevuto il 5 Giugno 1954)

Nell'esame statistico delle fluttuazioni dei numeri, sia di granuli singoli, sia di gruppi di granuli (costituiti da uno o più granuli susseguentisi senza soluzione di continuità: « blobs »), in celle uguali lungo tracce di particelle ionizzanti registrate in emulsioni nucleari, si è abitualmente fatto ricorso ad una delle seguenti schematizzazioni:

a) si ammette che la distribuzione degli eventi « punto iniziale di un granulo sviluppato » lungo la traccia, segua la statistica di Poisson, senza tenere conto della limitazione imposta alla posizione di un evento siffatto dalla distribuzione preventiva dei granuli di AgBr nell'emulsione; l'influenza delle dimensioni finite dei granuli sulla frequenza e sulle fluttuazioni è presa in considerazione seguendo i procedimenti statistici che si applicano ai conteggi con i contatori di Geiger e Müller <sup>(1)</sup>;

b) si tratta la distribuzione dei granuli sviluppati secondo la statistica di Bernoulli, come se ogni cella della traccia

sia suddivisa in un numero fisso  $N$  di posti, ciascuno dei quali è occupabile da un granulo sviluppato, con una certa probabilità  $p$  per l'evento « granulo sviluppato » <sup>(2)</sup>, trascurando sia il fatto che i granuli sviluppabili non sono rigorosamente a contatto, nè di eguali dimensioni, nè con i centri allineati (e quindi in numero non necessariamente costante in ogni cella della traccia), sia le variazioni delle dimensioni dei granuli durante lo sviluppo, e le conseguenze dello sviluppo indotto (questione trattata recentemente da DELLA CORTE, RAMAT e RONCHI <sup>(3)</sup>).

Nel caso di conteggi di granuli singoli, l'applicazione della statistica di Bernoulli ha fornito risultati in soddisfacente accordo con i dati sperimentali, per quanto riguarda le fluttuazioni statistiche, ed in particolare ha permesso di giustificare il fatto che lo scarto quadratico medio del numero di granuli, contato su celle uguali di una stessa traccia in un tratto a densità sensibilmente costante, è inferiore alla radice quadrata del valore me-

<sup>(1)</sup> P. E. HODGSON: *Journ. Appl. Phys.*, **3**, 11 (1952); L. JAUNEAU e F. HUGH-BOUSSER: *Journ. de Phys. et le Rad.*, **13**, 465 (1952); W. W. HAPP, T. E. HULL, A. H. MORRISH: *Canad. Journ. of Phys.*, **30**, 699 (1952). Altra bibliografia è contenuta in questi lavori.

<sup>(2)</sup> A. C. COATES: *Photographic Sensitivity* (London, 1951), p. 320; L. BARBANTI-SILVA, C. BONACINI, C. DE PIETRI, G. LOVERA, R. PERILLI FEDELI e A. ROVERI: *Atti Sem. Mat. Fis. dell'Un. di Modena*, **5**, 213 (1952).

<sup>(3)</sup> M. DELLA CORTE, M. RAMAT e L. RONCHI jr.: *Nuovo Cimento*, **10**, 958 (1953).

dio del numero di granuli in questione <sup>(2)</sup>. Infatti il numero medio  $v$  di granuli sviluppati ed il corrispondente scarto quadratico medio  $\sigma_v$  sono dati dalle formule

$$(1) \quad v = np$$

$$(2) \quad \sigma_v = [np(1-p)]^{\frac{1}{2}} = v^{\frac{1}{2}}(1-p)^{\frac{1}{2}},$$

dove  $n$  è il numero massimo di granuli sviluppati allineati, che la cella può contenere nel senso precisato al punto *b*), e  $p$  la probabilità che un granulo di bromuro d'argento, sulla traiettoria della particella ionizzante, dia luogo ad un granulo sviluppato.

Poichè lo stesso fatto si verifica nel conteggio dei gruppi di granuli <sup>(3-5)</sup>, per estendere il confronto, si è proceduto al calcolo dello scarto quadratico medio per questi conteggi, secondo la statistica di Bernoulli. I gruppi di granuli sono stati considerati come iterazioni dell'evento singolo « granulo sviluppato » <sup>(3)</sup>. Per lo scarto quadratico medio  $\sigma_a$  del numero di gruppi in celle uguali, si è ottenuta la seguente espressione (a meno di termini che diventano trascurabili al crescere della lunghezza della cella):

$$(3) \quad \sigma_a = a^{\frac{1}{2}} \left( 1 - 3 \frac{a}{n} \right) = a^{\frac{1}{2}} \left( 1 - 3 \frac{ad}{l} \right)^{\frac{1}{2}},$$

dove

$$(4) \quad a = np(1-p)$$

è il numero medio dei gruppi per ogni cella <sup>(3)</sup>, (anche qui a meno di termini trascurabili),  $d = l/n$  il diametro (medio) del granulo sviluppato,  $l$  la lunghezza della cella. Il numero medio  $a$  passa per un massimo piuttosto appiattito, per  $p=0,5$  se si assume  $n=\text{cost.}$

I particolari del calcolo saranno riportati in una nota successiva, insieme ai risultati di determinazioni sperimentali, attualmente in corso, aventi lo scopo di controllare la validità d'impiego della formula (3) nel caso di conteggio di granuli nelle emulsioni nucleari. Intanto, a titolo di esempio, si possono confrontare le formole in questione con alcuni dati numerici riportati da VOYVODIC <sup>(7)</sup>. In una duplice serie di conteggi, sia di granuli singoli, sia di gruppi di granuli, effettuati su 420 celle successive di 50  $\mu$ , lungo un tratto di 21 mm di una traccia relativistica, gli scarti quadratici medi sono risultati inferiori a quelli prevedibili per eventi casuali, secondo un fattore 0,92 per i conteggi di granuli singoli, e secondo un fattore 0,78 per i gruppi. I valori medi dei numeri di granuli singoli e di gruppi per cella (letti sulle curve di distribuzione di fig. 11 a pag. 257 del lavoro citato) sono, approssimativamente:  $v=13,5$  e  $a=11,5$ . Ne segue, dalle (1) e (4):  $p=1-a/v=0,148$ ;  $n=v/p=91,1$  (granuli per 50  $\mu$ ); sostituendo questi dati nelle (2) e (3), si ha:  $\sigma_v=0,92_3 v^{\frac{1}{2}}$ ;  $\sigma_a=0,78_8 a^{\frac{1}{2}}$ , in buon accordo con i fattori numerici sperimentali dati da VOYVODIC.

Analoghe concordanze sono state verificate con altri dati.

<sup>(4)</sup> R. R. DANIEL e D. H. PERKINS: *Proc. Roy. Soc.*, A **221**, 351 (1954).

<sup>(5)</sup> L. VOYVODIC: *Progress in Cosmic Ray Physics*, (Amsterdam, 1954), vol. II.

<sup>(6)</sup> R. VON MISES: *Zeits. f. angew. Math. u. Mech.*, **1**, 298 (1921).

<sup>(7)</sup> L. VOYVODIC: loc. cit., pag. 256-7.

# The $\pi^-/\pi^+$ Ratio from Deuterium Near Photo-Pion Threshold.

M. BENEVENTANO, D. CARLSON-LEE, G. STOPPINI

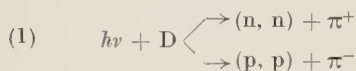
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(ricevuto il 9 Giugno 1954)

A systematic investigation of the photo-pion production from deuterium, i.e., of the reactions



is now in progress in collaboration with the Betatron Laboratory of the University of Illinois (Urbana).

A target of liquid  $D_2$  has been irradiated by the X-ray beam of the Illinois 300 MeV betatron. The mesons have been detected both by nuclear emulsion plates and stacks of pellicles (immersed in large blocks of emulsion). Several exposures have been made using photons of 195, 230, 245, 280 and 300 MeV maximum energy.

We present here, as a preliminary result, the value of the ratio  $f = \sigma(\pi^-)/\sigma(\pi^+)$  of the cross-sections of the reactions for the two energy-bins near the pion-threshold. This ratio is connected with some general features of the photo-pion matrix elements. The importance of a careful estimate of  $f$  as a function of the pion energy has been pointed out

by WATSON<sup>(1)</sup> and, more recently, by KROLL and RUDERMAN<sup>(2)</sup> and by WATSON<sup>(3)</sup>.

The experimental arrangement quite similar to that used by BERNARDINI and GOLDWASSER<sup>(4)</sup> is sketched in Fig. 1. It shows the essential simplicity sought

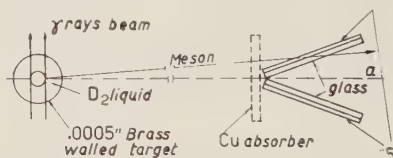


Fig. 1.

in the experiment. The target, already described in detail<sup>(5)</sup>, was a cylinder  $1\frac{1}{4}$ " in diameter with brass-walls 0.0005" thick. The meson background due to

<sup>(1)</sup> K. M. WATSON: *Phys. Rev.*, **85**, 852 (1952).

<sup>(2)</sup> N. M. KROLL and M. A. RUDERMAN: *Phys. Rev.*, **93**, 239 (1954).

<sup>(3)</sup> R. M. WATSON (to be published in *Phys. Rev.*).

<sup>(4)</sup> G. BERNARDINI and E. L. GOLDWASSER: *Phys. Rev.*, **94**, 729 (1954).

<sup>(5)</sup> E. A. WHALIN and R. A. REITZ (to be published in *Rev. of Scient. Instr.*).



these walls was found to be less than 3%. The plates were placed at the various angles, ranging between 30 and 150 degrees as is shown in fig. 2. Pions emerging from the target were able to reach the surface of the emulsions through the 0.017" Al foils covering the large

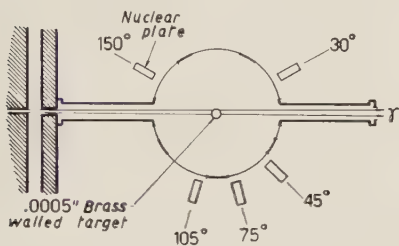


Fig. 2.

windows opened in the vacuum shell of the target.

In the first two exposures (maximum photon-energies 195 and 245 MeV respectively) two twin plates were mounted at each angle so that mesons entered into the emulsion at a glancing angle of about 5°. The plate holders, made out of polystyrene foam, were equipped with windows closed by 0.0005" Al foils. The distance from the center of the target to the center of the plates was  $10\frac{3}{4}$ ".

In the other two exposures (maximum photon energy, 300 MeV) the plates, in a group of 4, were placed at 105° with respect to the beam direction and rotated around the  $a$  axis (see Fig. 2); in one of these exposures, the plates were preceded by a copper absorber of 9.23 g/cm<sup>2</sup>.

In all experiments, the plates were Ilford G5 nuclear emulsions of 600  $\mu$  thickness with a total area of  $1.0 \times 3.0$  in<sup>2</sup>.

In the scanning of the plates, only those tracks having an entrance angle consistent with having come directly from the target were classified. The percentage of mesons showing «scattering in the walls» was less than 5%.

Only tracks longer than 500  $\mu$  were considered; those which ended within  $< 20$   $\mu$  from either surface of the emuls-

ion were also neglected. The sign of the mesons was decided from the characteristics of the end points of the tracks. With regard to this, the following phenomenological classification was adopted:

- $\rho$ : tracks of mesons having no heavily ionizing secondary longer than 6  $\mu$ .
- $\sigma_1$ : tracks with only one heavily ionizing secondary longer than 6  $\mu$  (surely  $\pi^-$ ).
- $\sigma_n (n \geq 2)$ : tracks with two or more heavily ionizing secondaries longer than 6  $\mu$  (surely  $\pi^-$ ).
- $\pi$ : tracks having a secondary of mesonic nature (surely  $\pi^+$ ).
- $\sigma$ : tracks having a single secondary whose identity as a  $\mu$ -meson or proton could not be established.

Evidently the  $\rho$  class contains that fraction of  $\pi^-$ -mesons which are captured and give no visible prongs ( $\sigma_0$ ) and a fraction of  $\mu$ -mesons produced by the decay in flight of the  $\pi^\pm$  before arrival at the plates. The  $\sigma$  class contains  $\pi$ - $\mu$  ( $\pi^+$ ) and  $\sigma_1$  ( $\pi^-$ ).

The plates from the first two exposure were underdeveloped in order to eliminate part of the electrons produced by the large number of photons and electrons in the neighbourhood of the betatron. This made it impossible to observe the decay electrons of the mesons and thus to separate the  $\sigma_0$ 's in the  $\rho$  class.

In addition, the presence of the remaining background did not allow the use of a definite criterion in the separation of plus and minus mesons of the class  $\sigma$ . For these plates the calculation of the  $\pi^-$  totals (and, by subtraction, of the  $\pi^+$  totals) has been based on the class  $\sigma_n (n \geq 2)$  and on the percentages  $A_0, A_1, \dots$ , of negative mesons, which, brought to rest in the emulsion give 0 visible prongs, 1 visible prong, etc.. The determination of the coefficients  $A_0, A_1, \dots$ , has been carried out on 1634  $\pi^-$ -mesons, using Ilford G5 600  $\mu$  plates

exposed to a beam of magnetically focussed mesons produced by protons on a carbon target (we are very much indebted to Dr. J. O'REAR, University of Chicago, who gave us these plates). The convention of classifying only recoils longer than  $6\mu$  as prongs was adopted for this determination, and the values

the method followed for the separation of the mesons of the two signs.

Both methods are susceptible to various controls, and the checks of internal consistency turn out to be very satisfactory. In Table II, we report the values of the ratio  $f = \sigma(\pi^-)/\sigma(\pi^+)$  for various angles between the gamma beam

TABLE I.

$A_0$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$
$30.2 \pm 2.1$	$32.4 \pm 1.3$	$18.4 \pm 1.0$	$12.8 \pm .8$	$5.5 \pm .6$	$1.0 \pm .2$
$30.5 \pm 1.5$	$31.6 \pm 2.0$	$19.4 \pm 1.5$	$12.9 \pm 1.3$	$5.0 \pm .8$	$0.6 \pm .3$

obtained are reported in the first line of Table I.

The two successive exposures relative to process (1) ( $E_{\gamma \max} = 300$  MeV) were carried out under conditions in which the background of electrons and photons around the betatron was strongly reduced so that normal development of the plates was possible. In this case, the visibility of the decay electrons of the  $\mu$ -mesons allowed a separation of the  $\sigma_0$ 's from the  $\mu$  contained in the  $\rho$  class. The percentage of negative  $\mu$ 's absorbed by nuclei and not decaying into an electron has been evaluated using the value given by FRY<sup>(6)</sup>. Furthermore, the relatively low background has made acceptable the criterium of assigning all those  $\sigma$  tracks which present a club or recoil to the  $\sigma\pi$  class and the remaining to the  $\pi\mu$  class. Thus, the separation of the mesons of the two signs has been carried out without recourse to the coefficients  $A_n$ . Instead, in this way, we obtained an independent determination of these coefficients which is reported in the second line of Table I. This furnished an excellent control of

and the direction of emission of the  $\pi$ -mesons. The errors reported have been calculated keeping in mind the instrumental errors as well as the statistical

TABLE II.

Angle (Lab.)	$\bar{E}_p$	$\bar{E}_\gamma$	$f = \frac{\sigma(\pi^-)}{\sigma(\pi^+)}$	No. of tracks
105	10.0	230	$1.40 \pm 0.12$	612
105	11.5	180	$1.51 \pm .11$	1023
75	18.2	176	$1.62 \pm .24$	323
135	18.9	195	$1.48 \pm .30$	177
150	17.7	196	$1.56 \pm .46$	61

errors. Column 2 gives the average energy of the ending mesons corresponding to the various determinations.

In column 3 the corresponding photon energies are presented assuming the nucleons were initially at rest. It was felt that the nucleon motion would not change these photon averaged energies too much, because near the threshold the photo-meson cross-section does not depend critically upon the photon-energy.

(<sup>6</sup>) W. F. FRY: *Nuovo Cimento*, **10**, 490 (1953).

Previous measurements <sup>(7)</sup> of  $f$  gave results which seemed to differ appreciably from those presented in Table II. It is felt that the differences are mainly due to the differences in the energies of the photon bins and to some corrections (nuclear absorptions, scatterings at the walls, etc.) needed by these previous measurements <sup>(7)</sup>.

Furthermore the use of a definite classification of the  $A$ 's coefficients insures a better consistency.

It is worthwhile to mention that a value of  $f$ , sensibly different from 1 is expected because of the nucleon recoil effects. As a matter of fact, the matrix element of the  $S$ -wave, which predomi-

nates for photon energies below 200 MeV, includes the electric dipole strength in the final state (<sup>3,8</sup>). This dipole effect does not act on the  $P$ -wave which can be originated, by conservation principles, only by magnetic dipole or electric quadrupole absorption.

This experiment will be fully discussed in a forthcoming extensive paper.

The authors wish to thank sincerely Prof. A. O. HANSON and Mr. H. R. REITZ for their kind cooperation in the measurements of the spectrum and the construction of the liquid targets. Particular thanks are due to Dr. L. TAU for valuable help in the collection and elaboration of the data.

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<sup>(7)</sup> R. M. LITTAUER and D. WALKER: *Phys. Rev.*, **82**, 746 (1951); L. LEBOW, B. T. FELD, D. H. FRISLIK and L. S. OSBORNE: *Phys. Rev.*, **85**, 681 (1952); R. S. WHITE, M. J. JACOBSON and A. G. SCHULZ: *Phys. Rev.*, **88**, 836 (1952).

<sup>(8)</sup> G. BERNARDINI and E. L. GOLDWASSER: (to be published in *Phys. Rev.*).

## Watson's Type Relations for $\Lambda$ -Particle Production.

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It was shown by WATSON <sup>(1)</sup>, using an argument based on group theory, that, due to charge independence, a general theorem holds in multiple pion production in nucleon-nucleon collisions, at any order of multiplicity, which poses a restriction on the relative number of charged and neutral pions created. Later VAN HOVE <sup>(2)</sup> demonstrated that the same theorem follows from much simpler arguments. It is very simple to formulate relations of the kind found by WATSON for processes which lead to the production of new unstable particles (heavy mesons, hyperons). Of course no *a priori* reasons can at present be given to support the view that charge independence should hold also in processes where new particles are produced. However, in order to draw any conclusions on this question in the future, it seems of interest to point out the most simple and general consequences which would follow from charge independence.

Starting from the relations (6) of a previous note <sup>(3)</sup>, one may easily obtain

<sup>(1)</sup> K. M. WATSON: *Phys. Rev.*, **85**, 852 (1952).

<sup>(2)</sup> L. VAN HOVE: *Rochester Report* NYO 3074; see also L. VAN HOVE, R. E. MARSHAK and A. PAIS: *Phys. Rev.*, **88**, 1211 (1952).

<sup>(3)</sup> R. GATTO: *Nuovo Cimento*, **11**, 445 (1954).

the relation

$$\begin{aligned} & \sigma(pp|\Lambda^{++}\Lambda^0) + \sigma(pp|\Lambda^0\Lambda^{++}) + \\ & + 2[\sigma(pn|\Lambda^{++}\Lambda^-) + \sigma(pn|\Lambda^-\Lambda^{++})] = \\ & = \sigma(pp|\Lambda^{++}\Lambda^0) + \sigma(pp|\Lambda^0\Lambda^{++}) + \\ & + 2[\sigma(pn|\Lambda^+\Lambda^0) + \sigma(pp|\Lambda^+\Lambda^+) + \\ & + \sigma(pn|\Lambda^0\Lambda^+)]. \end{aligned}$$

To put it in a simpler form imagine a statistical ensemble of nucleon-nucleon collisions <sup>(2)</sup>: the incident particle is always a proton in a *well-defined* state *a* (for instance with assigned momentum and assigned polarization), the target nucleon is a proton or a neutron, with equal probabilities, in a *well defined* state *b*. The above relation may then be written in the form

$$(1) \quad N_{++} + N_- = N_+ + N_0,$$

where we call  $N_{++}$ ;  $N_+$ ;  $N_0$  and  $N_-$  respectively the average number of doubly charged, singly charged, neutral and negatively charged  $\Lambda$ -particles which emerge after the collision in the two *well-defined* states *c* and *c'*. The various relations of the form (1), which hold for the different polarizations of the initial and final particles, can be summed, giving a



relation which has still the same form as (1). Thus one can say that relation (1) still holds for the unpolarized cross-sections. Moreover relation (1) still holds for the total cross-sections, since it is easy to see that the possible indistinguishability of the final particles (when they have the same charge) is correctly accounted for by the definition adopted for the numbers  $N_{++}$ ;  $N_+$ ;  $N_0$  and  $N_-$ .

The validity of (1) is not restricted to the particular production process considered above: it follows quite generally from charge independence alone, as long as charge independence is assumed. Thus it holds for nucleon-nucleon collisions in which  $\Lambda$ -particles emerge (singly or paired) together with any kind and any number of other particles, as long as charge independence is not violated in the process; *nor it is necessary that  $\Lambda$ -particles be produced directly, the only restriction being that charge independence be not violated for the whole process in any of the intermediate steps.*

For completeness we briefly report the very simple argument. Imagine the same statistical ensemble of collisions previously considered, but assume this time that the incident nucleon is always a neutron. Charge symmetry requires  $N_{++} = N_-$ ;  $N_+ = N_0$ ; where the  $N$ 's denote the average numbers of  $\Lambda$ -particles which now are produced. But for a larger statistical ensemble of collisions, where the incident particle has now the same probability of being a proton or a neutron, the relation  $N_{++} + N_+ = N_+ + N_+$  would follow from the initial isotropy in charge space, which charge independence preserves; so that (1) follows. Of course, relation (1) remains valid if the incident particle is a neutron instead of being a proton.

For the case when the incident particle is a pion the relation

(where we call  $N_{++}^+$  the average number of  $\Lambda^{++}$  produced when the incident pion is positive, the other  $N$ 's being similarly defined) can be derived. Thus consider for instance the process recently discovered by FOWLER, SHUTT, THORNDIKE, WHITTMORE<sup>(4)</sup>, pion + nucleon  $\rightarrow \Lambda + \theta$  (and assume the isotopic spin of the  $\theta$ -particle to be unity; relation (2), of course, similarly holds for any other assigned isotopic spin of the  $\theta$ -particle). Relation (2) would be written:

$$\begin{aligned} & \sigma(\pi^+p|\Lambda^{++}\theta^0) + \sigma(\pi^+n|\Lambda^{++}\theta^-) - \\ & \quad - \sigma(\pi^+p|\Lambda^+\theta^+) - \sigma(\pi^+n|\Lambda^0\theta^+) - \\ & \quad - \sigma(\pi^+n|\Lambda^+\theta^0) = -\sigma(\pi^0p|\Lambda^{++}\theta^-) + \\ & \quad + \sigma(\pi^0p|\Lambda^+\theta^0) + \sigma(\pi^0n|\Lambda^+\theta^-). \end{aligned}$$

It is easy to see that in fact this last relation follows from the relations (4) given in a previous note<sup>(3)</sup>. Unfortunately in (2) the appearance of cross-sections referring to incident neutral pions cannot be avoided.

From the relationship (1) one would approximately infer that, if charge independence holds, when a nucleon is incident on a nucleus (which for our purposes may be regarded as having the same number of protons and neutrons) doubly charged and negatively charged  $\Lambda$ -particles should be produced with the same frequency as positively charged and neutral  $\Lambda$ -particles. [The production of  $\Lambda$ -particles could also be due to the collisions of pions, produced in the reaction, with the other constituent nucleons: this however does not spoil the validity of the argument, since such secondary collisions in the same nucleus take place in a time short compared to the pion lifetimes]. When the incident particle is a charged pion a similar conclusion cannot immediately be reached starting from the relationship (2). However it seems highly reasonable that a similar situation should occur<sup>(3)</sup>, although a deeper study of the

$$\begin{aligned} (2) \quad N_{++}^+ - N_+^+ - N_0^+ + N_-^+ &= \\ &= -N_{++}^0 + N_+^0 = N_0^0 - N_-^0 \end{aligned}$$

(4) W. B. FOWLER, R. P. SHUTT, A. M. THORNDIKE and W. L. WHITTMORE: *Phys. Rev.*, **93**, 861 (1954).



implications of charge independence in the particular case would be required.

It has been considered that  $\Lambda$ -particles may possibly possess unity isotopic spin <sup>(5)</sup> (only  $\Lambda^+$ ,  $\Lambda^0$ ,  $\Lambda^-$  should exist) and that charge independence holds for production (but not for decay). In this case one would expect Watson's relation ( $N_+ + N_- - 2N_0$ ) to hold for  $\Lambda$ -particle production in nucleon-nucleon collisions; for pion-nucleon collisions a relation of the form  $N_-^+ + N_+^+ = 2N_0^+ + N_0^0 - N_+^0$  should be valid, practically of little interest to us due to the appearance of cross-sections referring to incident neutral pions ( $N_0^0$  and  $N_+^0$ ).

Perhaps the experimental situation is still too immature for definite conclusions. However we believe that one should eventually be prepared, either to reject the hypothesis that charge independence holds for the production of  $\Lambda$ -particles,

or to consider that those particles which at present are called  $\Lambda^+$ ,  $\Lambda^0$ ,  $\Lambda^-$  (and the suggested  $\Lambda^{++}$ ) might not be different charge states of the same particle. The first eventuality seems to be in contrast with the tendency to regard these particles as states of the meson nucleon field, rather than as new elementary particles, and the contemporary assumption that charge independence holds in meson-nucleon processes; the second eventuality could perhaps be supported by the fact that the measured rest masses of  $\Lambda^0$  and  $\Lambda^+$  are in fact sensibly different, probably somewhat more than one would be inclined to accept as due to charge dependent interactions.

I would like to thank Prof. E. AMALDI and Prof. B. FERRETTI for their interest and encouragement. Moreover I wish to express my gratitude to Prof. R. E. MARSHAK for a helpful discussion on questions concerning charge independence.

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<sup>(5)</sup> M. GELL-MANN: *Phys. Rev.*, **93**, 833 (1953).

# ERRATA CORRIGE

C. DILWORTH, S. J. GOLDSACK and L. HIRSCHBERG: **Determination of the Mass of Slow Particles by the Constant Sagitta Method**, *Nuovo Cimento* **11**, 113 (1954).

Pag. 117: In all cases, in place of  $\lambda^{-1}$  read  $\overline{\lambda^{-1}}$ .

Pag. 123: For the formula

$$\overline{D} = D_s \sum_0^N (\delta_K \delta_{ER} \delta_{P\beta})^{-1} / N = \overline{\lambda^{-1}} D_s$$

read:

$$\overline{D} = D_s \sum_0^N (\delta_K \delta_{ER} \delta_{P\beta})^{-1} / N = D_s / \overline{\lambda^{-1}}.$$

Also in the 5th line from the end, in place of

$$\delta^{-1} = (\delta_K \delta_{ER} \delta_{P\beta})^{-1}$$

read

$$\lambda^{-1} = (\delta_K \delta_{ER} \delta_{P\beta})^{-1}.$$

In the Fig. 7 and in the 3rd line from the end in place of  $\lambda^{-1}$  read  $\overline{\lambda^{-1}}$ .

# ERRATA CORRIGE

G. BERTOLINI, A. BISI, F. LAZZARINI and L. ZAPPA: **On the L-Capture to K-Capture Ratio in Cd<sup>109</sup>**, *Nuovo Cimento*, **11**, 539 (1954).

On page 548, 24th line, instead of « ...  $Z_{\text{eff}} = Z - 0.3$  », read « ...  $Z_{\text{eff}} = Z - 0.3$  and  $Z_{\text{eff}} = Z - 4.15$  for K- and L-shell respectively ».

On page 550 the first equation should read:

$$q^* = \frac{2}{\cos^2 \varphi + \sin^2 \varphi \cos^2 \theta} \{ R^2 (\cos^2 \varphi + \cos^2 \theta \sin^2 \varphi) - D^2 \cos^2 \varphi (1 - \cos^2 \theta) \}^{\frac{1}{2}}.$$

PROPRIETÀ LETTERARIA RISERVATA